

# Turing Degree Spectra of Differentially Closed Fields

David Marker\* & Russell Miller†

June 17, 2014

## Abstract

The degree spectrum of a countable structure is the set of all Turing degrees of presentations of that structure. We show that every nonlow Turing degree lies in the spectrum of some differentially closed field (of characteristic 0, with a single derivation) whose spectrum does not contain the computable degree  $\mathbf{0}$ . Indeed, this is an equivalence, for we also show that every such field of low degree is isomorphic to a computable differential field. Relativizing the latter result and applying a theorem of Montalbán, Soskova, and Soskov, we conclude that the spectra of countable differentially closed fields of characteristic 0 are exactly the jump-preimages of spectra of automorphically nontrivial countable graphs.

## 1 Introduction

Differential fields arose originally in work of Ritt examining algebraic differential equations on manifolds over the complex numbers. Subsequent work by Ritt, Kolchin and others brought this study into the realm of algebra,

---

\*This work was initiated at a workshop held at the American Institute of Mathematics in August 2013, where the question of noncomputable differentially closed fields was raised by Wesley Calvert. The authors appreciate the support of A.I.M., and also thank Calvert and Hans Schoutens for useful conversations.

†Partially supported by Grant # DMS – 1001306 from the National Science Foundation, and by grant # 66582-00 44 from The City University of New York PSC-CUNY Research Award Program.

where numerous parallels appeared with algebraic geometry. The topic first intersected with model theory in the mid-twentieth century, in work of Abraham Robinson, and logicians soon discovered the theories of differential fields and of differentially closed fields to have properties which had been considered in the abstract, but had not previously been known to hold for any everyday theories in mathematics. It was the model theorists who provided the definitive resolution to the question of differential closure, several variations of which had previously been developed in differential algebra. In 1974, Harrington proved the existence of computable differentially closed fields, making the notion more concrete, although our grasp of this topic remains more tenuous than our understanding of algebraic closures in field theory.

In this article, we offer an analysis of the complexity of countable differentially closed fields of characteristic 0. This work requires a solid background in differential algebra, in model theory, and in effective mathematics. Ultimately we will characterize the spectra of countable models of  $\mathbf{DCF}_0$  (the theory of differentially closed fields of characteristic 0) as exactly the preimages, under the jump operation, of spectra of automorphically nontrivial countable graphs; or, equivalently, as exactly those spectra of such graphs which are closed under a simple equivalence relation on Turing degrees. To do so, we show that spectra of differentially closed fields have certain complexity properties, which are not known to hold of any other standard class of mathematical structures: every low differentially closed field of characteristic 0 is isomorphic to a computable one, whereas every nonlow degree computes a differentially closed field which has no computable copy. Indeed we will present a substantial class of fairly complex spectra that can all be realized by models of  $\mathbf{DCF}_0$ , including spectra with arbitrary proper  $\alpha$ -th jump degrees, for every computable nonzero ordinal  $\alpha$ . To explain what these results mean, we begin immediately with the necessary background. For supplemental information on computability theory, [27] is a standard source, while for more detail about model theory and differential fields, we suggest [15], [20], or the earlier [24].

## 1.1 Background in Differential Algebra

A differential ring is a ring with a *differential operator*, or *derivation*, on its elements. If the ring is a field, we call it a *differential field*. The differential operator  $\delta$  is required to preserve addition and to satisfy the familiar *Leibniz Rule*:  $\delta(x \cdot y) = (x \cdot \delta y) + (y \cdot \delta x)$ . Examples include the field  $\mathbb{Q}(x)$  of rational

functions over  $\mathbb{Q}$  in a single variable  $x$ , with the usual differentiation  $\frac{d}{dx}$ , or the field  $\mathbb{Q}(t, \delta t, \delta^2 t, \dots)$ , with  $\delta$  acting as suggested by the notation. In these examples,  $\mathbb{Q}$  may be replaced by another differential field  $K$ , with the derivation  $\delta$  on  $K$  likewise extended to all of  $K(x)$  or  $K(t, \delta t, \dots)$ . (The only possible derivation on  $\mathbb{Q}$  maps all rationals to 0. In general, the *constants* of a differential field  $K$  are those  $x \in K$  with  $\delta x = 0$ , and they form the *constant subfield*  $C_K$  of  $K$ .) We use angle brackets and write  $K\langle y_i : i \in I \rangle$  for the smallest differential subfield (of a given extension of  $K$ ) containing all the elements  $y_i$ ; this is well-defined, and this subfield is said to be *generated* as a differential field by  $\{y_i \mid i \in I\}$ . Of course, the field generated by these same elements may well be a proper subfield of this: in the examples above,  $\mathbb{Q}\langle x \rangle = \mathbb{Q}(x)$ , but  $\mathbb{Q}(t) \subsetneq \mathbb{Q}\langle t \rangle = \mathbb{Q}(t, \delta t, \dots)$ . Differentiation of rational functions turns out to follow the usual quotient rule, bearing in mind that  $\delta$  may well map coefficients in a nonconstant ground field  $K$  to elements other than 0.

For the purposes of this article, we restrict ourselves to characteristic 0 and to *ordinary* differential rings and fields, i.e., those with only one derivation. *Partial* differential rings, with more differential operators, exist and have natural examples, as do differential rings of positive characteristic, but considering either would expand this article well beyond the scope we intend.

For a differential ring  $K$  with derivation  $\delta$ ,  $K\{Y\}$  denotes the ring of all *differential polynomials* over  $K$ ; it may be viewed as the ring of algebraic polynomials  $K[Y, \delta Y, \delta^2 Y, \dots]$ , with  $Y$  and all its derivatives treated as separate variables. (One sometimes differentiates a differential polynomial, treating each  $\delta^{n+1}Y$  as the derivative of  $\delta^n Y$ .) Iterating this,  $K\{Y_0, \dots, Y_{n+1}\}$  is defined as  $(K\{Y_0, \dots, Y_n\})\{Y_{n+1}\}$ . With only one derivation in the language, we often write  $Y'$  for  $\delta Y$ , or  $Y^{(r)}$  for  $\delta^r Y$ .

The *order* of a nonzero differential polynomial  $q \in K\{Y\}$  is the greatest  $r$  such that the  $r$ -th derivative  $Y^{(r)}$  appears nontrivially in  $q$ . Equivalently, it is the least  $r$  such that  $q \in K[Y, Y', \dots, Y^{(r)}]$ . Having order 0 means that  $q$  is an algebraic polynomial in  $Y$  of degree  $> 0$ ; nonzero elements of  $K$  within  $K\{Y\}$  are said to have order  $-1$ . Each polynomial in  $K\{Y\}$  also has a *rank* in  $Y$ . For two such polynomials, the one with lesser order has lesser rank. If they have the same order  $r$ , then the one of lower degree in  $Y^{(r)}$  has lesser rank. Having the same order  $r$  and the same degree in  $Y^{(r)}$  is sufficient to allow us to reduce one of them, modulo the other, to a polynomial of lower degree in  $Y^{(r)}$ , and hence of lower rank: just take an appropriate  $K$ -linear combination of the two. So, for our purposes, the rank in  $Y$  is simply given by

the order  $r$  and the degree of  $Y^{(r)}$ . Therefore, all ranks of nonzero differential polynomials are ordinals in  $\omega^2$ .

Our convention in this article is that the zero polynomial has order  $+\infty$ . Thus, for every element  $x$  in any differential field extension of  $K$ , the *minimal differential polynomial* of  $x$  over  $K$  is defined (up to a nonzero scalar from  $K$ ) as the differential polynomial  $q$  in  $K\{Y\}$  of least rank for which  $x$  is a zero (i.e.,  $q(x) = 0$ ). In particular, the zero polynomial is considered to be the minimal differential polynomial of an element differentially transcendental over  $K$  (such as  $t$  in  $\mathbb{Q}\langle t \rangle$  above); this is simply for notational convenience.

The *differential closure*  $\widehat{K}$  of a differential field  $K$  is the prime model of the theory  $\mathbf{DCF}_0 \cup \Delta(K)$ , the union of the atomic diagram  $\Delta(K)$  of  $K$  with the (complete, decidable) theory  $\mathbf{DCF}_0$  of differentially closed fields of characteristic 0. This theory was effectively axiomatized by Blum (see [2] or [3]): her axiom set for a differentially closed field  $F$  includes the axioms for differential fields of characteristic 0 and states that, for each pair  $(p, q)$  of differential polynomials with arbitrary coefficients from  $F$  and with  $\text{ord}(p) > \text{ord}(q)$ , the axiom that  $F$  contains an element  $x$  with  $p(x) = 0 \neq q(x)$ . (By our convention on ranks,  $\text{ord}(p) > \text{ord}(q)$  ensures that  $q$  is not the zero polynomial, but does allow  $p$  to be an algebraic polynomial in  $F[Y]$  if  $q$  is a nonzero constant. Thus  $F$  must be algebraically closed.)

Blum proved  $\mathbf{DCF}_0$  to be  $\omega$ -stable, and existing results of Morley then showed that the theory  $\mathbf{DCF}_0 \cup \Delta(K)$  always has a prime model, i.e., every differential field  $K$  has a differential closure. Subsequently, Shelah established that, as the prime model extension of an  $\omega$ -stable theory, the differential closure  $\widehat{K}$  of  $K$  is unique and realizes exactly those types principal over it. Each principal 1-type has as generator a formula of the form  $p(Y) = 0 \neq q(Y)$ , where  $(p, q) \in (K\{Y\})^2$  is a *constrained pair*. By definition, this means that  $p(Y)$  is a monic, algebraically irreducible polynomial in  $K\{Y\}$ , that  $q$  has strictly lower rank in  $Y$  than  $p$  does, and that, in  $\widehat{K}$  (and hence in every differential field extension of  $K$ ), every  $y$  satisfying  $p(y) = 0 \neq q(y)$  has minimal differential polynomial  $p$  over  $K$ . (A fuller definition appears in [16, Defn. 4.3].) Hence the elements satisfying the generating formula form an orbit under the action of those automorphisms of  $\widehat{K}$  that fix  $K$  pointwise. For a pair  $(p, q)$  to be constrained is a  $\Pi_1^K$  property, and there exist computable differential fields  $K$  for which it is  $\Pi_1$ -complete. If such a  $q$  exists, then  $p$  is said to be *constrainable*; clearly this property is  $\Sigma_2^K$ . Not all monic irreducible polynomials in  $K\{Y\}$  are constrainable: for example,  $\delta Y$  is not.

More generally, no  $p$  in the image of  $K\{Y\}$  under  $\delta$  is constrainable, and certain polynomials  $p$  outside this image are also known to be unconstrainable. In fact, constrainability has been shown in [16] to be  $\Sigma_2^0$ -complete for certain computable differential fields  $K$ . The complexity of constrainability over the constant differential field  $\mathbb{Q}$  is unknown: it could be as low as  $\Delta_1^0$  or as high as  $\Sigma_2^0$ . We note that  $p$  is constrainable over  $K$  if and only if some  $y \in \widehat{K}$  has minimal differential polynomial  $p$  over  $K$ . (This equivalence will be extremely useful in the  $\mathcal{S}_m$ -substages of the construction for Theorem 4.1.) In order for this argument to show that constrainability is  $\Sigma_2^K$ , we need a computable presentation of  $\widehat{K}$ . This is provided by the following theorem, which will also be essential to our work in this paper for other reasons.

**Theorem 1.1 (Harrington; [7], Corollary 3)** *For every computable differential field  $K$ , there exists a differentially closed computable differential field  $L$  and a computable differential field homomorphism  $g : K \rightarrow L$  such that  $L$  is constrained over the image  $g(K)$ . Moreover, indices for  $g$  and  $L$  may be found uniformly in an index for  $K$ .*

So this  $L$  is in fact a differential closure of  $K$  – or at least, of the image  $g(K)$ , which is computably isomorphic to  $K$  via  $g$ . To preserve standard terminology, we continue to refer to the computable function  $g$  in Theorem 1.1 as a *Rabin embedding* for the differential field  $K$ . This is the term used for computable fields and computable presentations of their algebraic closures, the context in which Rabin proved the original analogue of this theorem. We note that the exposition in [7] does not consider uniformity of the procedure it describes, but a close reading of the proof there indicates that the algorithm giving  $g$  and  $L$  is indeed uniform in an index for the original computable differential field.

## 1.2 Background in Model Theory

Theorem 3.1 will require some background beyond Subsection 1.1, which we provide here, referring the reader to [15] and [21] (which are two chapters in the same volume) for details and further references regarding these results. Model theorists have made dramatic inroads in the study of differential fields and  $\mathbf{DCF}_0$ ; here we restrict ourselves to describing the results necessary to prove Theorem 3.1, without giving complete definitions of all the relevant concepts.

Let  $K$  be a differentially closed field, with subfield  $C_K$  of constants. For  $a \in K \setminus C_K$ , consider the elliptic curve  $E_a$  given by

$$y^2 = x(x-1)(x-a).$$

Let  $E_a^\sharp$  be the Kolchin closure of the set of all torsion points in the usual group structure on  $E_a$ . (The Kolchin topology is the differential analogue of the Zariski topology.) The set  $E_a^\sharp$  is known as the *Manin kernel* of this abelian variety, as it is the kernel of a certain homomorphism of differential algebraic groups. One construction of Manin kernels appears in [14]. In the proof of Theorem 3.1 we will use Manin kernels  $E_{a_m a_n}^\sharp$ , meaning  $E_a^\sharp$  as above with  $a = a_m + a_n$ .

**Theorem 1.2** *The family  $(E_a^\sharp : a' \neq 0)$  is definable. Indeed, it can be defined, uniformly in each  $a \in I$ , by a quantifier-free formula.*

The definability was claimed in [9] but is done more clearly in Section 2.4 of [19]. Of course, quantifier elimination for  $\mathbf{DCF}_0$  allows us to take the definition to be quantifier-free.

**Theorem 1.3** 1. *If  $a' \neq 0$ , then  $E_a^\sharp$  is strongly minimal and locally modular.*

2.  *$E_a^\sharp$  and  $E_b^\sharp$  are orthogonal if and only if  $E_a$  and  $E_b$  are isogenous. In particular if  $a$  and  $b$  are algebraically independent over  $\mathbb{Q}$ , then  $E_a^\sharp$  and  $E_b^\sharp$  are orthogonal.*

These results are due to Hrushovski and Sokolović [10], whose manuscript was never published. A proof of (1) is given in Section 5 of [14], and proofs of both (1) and (2) appear in Section 4 of [21].

**Corollary 1.4** *For every element  $(b_0, b_1)$  of  $E^\sharp(a)$  in the differential closure of  $\mathbb{Q}\langle a \rangle$ , both  $b_0$  and  $b_1$  are algebraic over  $\mathbb{Q}\langle a \rangle$ .*

*Proof.* Let  $\psi(b_0, b_1)$  be the formula over  $\mathbb{Q}\langle a \rangle$  isolating the type of  $(b_0, b_1)$ . If  $\psi$  defined an infinite subset of  $E_a^\sharp$ , then it would contain a torsion point. But if  $\psi$  contains an  $n$ -torsion point, every point in  $\psi$  would be an  $n$ -torsion point, yet there are only  $n^2$   $n$ -torsion points in  $E_a$ , a contradiction. Thus  $\psi(b_0, b_1)$  defines a finite set, so this pair is model-theoretically algebraic over  $a$ , hence lies in the field-theoretic algebraic closure of  $\mathbb{Q}\langle a \rangle$ . ■

**Lemma 1.5** *Let  $X$  and  $Y$  be strongly minimal sets defined over a differentially closed field  $K$ . If  $X$  and  $Y$  are orthogonal, then for any new element  $x \in X$  the differential closure of  $K\langle x \rangle$  contains no new elements of  $Y$ .*

Lemma 1.5 appears as [15, 7.2], while Lemma 1.6 can be found in Section 6 of [15].

**Lemma 1.6** *Let  $K$  be a differentially closed field and*

$$I = \{y \in K : y \neq 0 \text{ \& } y \neq 1 \text{ \& } y' = y^3 - y^2\}.$$

*Then  $I$  is a strongly minimal set of indiscernibles.*

Note that  $I$  must be a trivial strongly minimal set and hence  $I$  is orthogonal to each of the sets  $E_a^\sharp$ . (Also, the set  $I$  is computable in the Turing degree of the differential field  $K$ , as defined in the next subsection.)

**Lemma 1.7** *If  $a, b, c, d, \in I$ ,  $a \neq b$ ,  $c \neq d$  and  $\{a, b\} \neq \{c, d\}$ , then  $a + b$  and  $c + d$  are algebraically independent.*

*Proof.* Suppose  $p(X, Y) \in \mathbb{Q}[X, Y]$  such that  $p(a + b, c + d) = 0$ . There are only finitely many  $y$  such that  $p(a + b, y) = 0$ . Suppose without loss of generality that  $d \notin \{a, b\}$ . Then by indiscernibility  $p(a + b, c + e) = 0$  for every  $e \in I \setminus \{a, b, c\}$ , a contradiction. ■

### 1.3 Background in Computable Model Theory

Now we describe the concepts from computable model theory relevant to our work. For Theorems 3.1 and 4.1, only Definition 1.8 is really necessary, but the rest of the subsection will make clear why the broad results in Section 5 are of interest.

Let  $\mathcal{S}$  be a first-order structure on the domain  $\omega$ , in a computable language (e.g., any language with finitely many function and relation symbols). The (*Turing*) *degree*  $\deg(\mathcal{S})$  is the Turing degree of the atomic diagram of  $\mathcal{S}$ ; in a finite language, this is the join of the degrees of the functions and relations in  $\mathcal{S}$ .  $\mathcal{S}$  is *computable* if this degree is the computable degree  $\mathbf{0}$ . A structure isomorphic to a computable structure is said to be *computably presentable*; many countable structures fail to be computably presentable. A more exact measure of the presentability of (the isomorphism type of) the structure is given by its *Turing degree spectrum*.

**Definition 1.8** The *spectrum* of a countable structure  $\mathcal{S}$  is the set

$$\{\deg(\mathfrak{M}) : \mathfrak{M} \cong \mathcal{S} \ \& \ \text{dom}(\mathfrak{M}) = \omega\}$$

of all Turing degrees of *copies*  $\mathfrak{M}$  of  $\mathcal{S}$ .

Requiring that  $\text{dom}(\mathfrak{M}) = \omega$  prohibits complexity from being coded into the domain of the structure: the spectrum is intended to measure the complexity of the functions and relations, unaugmented by any trickery in choosing the domain. When dealing with fields, however, we often write  $\{x_0, x_1, \dots\}$  for the domain; otherwise the element 1 in  $\omega$  might easily be confused with the multiplicative identity in the field, for instance. In [12], Knight proved that spectra are always closed upwards, except in a few very trivial cases (such as the complete graph on countably many vertices, whose spectrum is  $\{\mathbf{0}\}$ ).

A wide range of theorems is known about the possible spectra of specific classes of countable structures. Many classes, including directed graphs, undirected graphs, partial orders, lattices, nilpotent groups (see [8] for all these results), and fields (see [17]), are known to realize all possible spectra. We will use the following specific theorem of Hirschfeldt, Khoussainov, Shore, and Slinko.

**Theorem 1.9** (see Theorem 1.22 in [8]) *For every countable, automorphically nontrivial structure  $\mathfrak{M}$  in any computable language, there exists a (symmetric, irreflexive) graph with the same spectrum as  $\mathfrak{M}$ .*

Richter showed in [22] that linear orders, trees and Boolean algebras fail to realize any spectrum containing a least degree under Turing reducibility, except when that least degree is  $\mathbf{0}$ , whereas undirected graphs can realize all such spectra. Boolean algebras were then distinguished from these other two classes when Downey and Jockusch showed that every low Boolean algebra has the degree  $\mathbf{0}$  in its spectrum; this has subsequently been extended as far as  $\text{low}_4$  Boolean algebras, in [4, 13, 29]. In contrast, Jockusch and Soare showed in [11] that each low degree does lie in the spectrum of some linear order with no computable presentation, although it remains open whether there is a single linear order whose spectrum contains all these degrees but not  $\mathbf{0}$ . (There does exist a graph whose spectrum contains all degrees except  $\mathbf{0}$ , by results in [26, 30]. A useful survey of related results appears in [6].)

Of relevance to our investigations are the algebraically closed fields of characteristic 0, which are the models of the closely related theory  $\mathbf{ACF}_0$ .



Here, however, the spectrum question has long been settled: every countable algebraically closed field has every Turing degree in its spectrum. On the other hand, every field becomes a constant differential field when given the zero derivation, which adds no computational complexity, and so the result mentioned above for fields shows that every possible spectrum is the spectrum of a differential field. These bounds leave a wide range of possibilities for spectra of differentially closed fields, and this is the subject of the present paper. It should be noted that, although every differentially closed field  $K$  is also algebraically closed and therefore is isomorphic (as a field) to a computable field, it may be impossible to add a computable derivation to the computable field in such a way as to make it isomorphic (as a differential field) to  $K$ .

We will show in Theorem 3.1 that countable differentially closed fields do realize a substantial number of quite nontrivial spectra, derived in a straightforward way from the spectra of undirected graphs. In particular, differentially closed fields can have all possible proper  $\alpha$ -th jump degrees (as defined in that section), for all computable ordinals  $\alpha > 0$ . Section 2 is devoted to general background material for the proof of Theorem 3.1. On the other hand, in Section 4, we prove Theorem 4.1, paralleling the original Downey-Jockusch result: it shows that if the spectrum of a countable differentially closed field contains a low degree, then it must also contain the degree  $\mathbf{0}$ .  $\mathbf{DCF}_0$  thus becomes the second theory known to have this property (apart from trivial examples such as  $\mathbf{ACF}_0$ ). Our positive results in the earlier section, however, show that this theorem does not extend to  $\text{low}_2$  degrees, let alone to  $\text{low}_4$  degrees, as holds for Boolean algebras. Thus  $\mathbf{DCF}_0$  realizes a collection of spectra not currently known to be realized by the models of any other theory in everyday mathematics. Finally, in Section 5, we relativize Theorem 4.1 and combine it with the results from Section 3 to characterize the spectra of models of  $\mathbf{DCF}_0$  precisely as the preimages under the jump operation of the spectra of automorphically nontrivial graphs, and also as those spectra of such graphs which have the particular property of being closed under first-jump equivalence.

## 2 Eventually Non-isolated Types

The model-theoretic basis of Theorem 3.1 is ENI-DOP, the *Eventually Non-Isolated Dimension Order Property*, developed by Shelah [25] in proving

Vaught's Conjecture for  $\omega$ -stable theories. In this section we give a simple example of how this property can be used to code graphs into models of theories satisfying ENI-DOP. The example may help the reader understand the coding in Section 3, which is another example of the same phenomenon, but which uses models of  $\mathbf{DCF}_0$  and hence is not so simple.

In our simple example, we have a language with two sorts  $A$  and  $F$ , and three unary function symbols  $\pi_1 : A \rightarrow F$ ,  $\pi_2 : A \rightarrow F$ , and  $S : F \rightarrow F$ . Our theory  $T$  includes axioms saying that  $A$  is infinite, that the map  $(\pi_1, \pi_2) : F \rightarrow A^2$  is onto, that  $\pi_i \circ S = \pi_i$ , and that  $S$  is a bijection from  $F$  to itself with no cycles.

This  $T$  is complete and has quantifier elimination. Its prime model consists of a countable set  $A$  with one  $\mathbb{Z}$ -chain  $F_{ab}$  (under  $S$ ) in  $F$  for each pair  $(a, b) \in A^2$ . (Here  $F_{ab}$  is the preimage of  $(a, b)$  under the map  $(\pi_1, \pi_2)$ , and is called the *fiber above*  $(a, b)$ .) It is clear that every permutation of  $A$  extends to an automorphism of the prime model, and so  $A$  is a set of indiscernibles, in this model and also in every other model of  $T$ .

The type over  $a$  and  $b$  of a single element  $x$  of the fiber  $F_{ab}$  is isolated by the formula  $(\pi_1(x) = a \ \& \ \pi_2(x) = b)$ . However, over one realization  $c$  of this type, the type of a new element of  $F_{ab}$  (not in the  $\mathbb{Z}$ -chain of  $c$ ) over  $a$ ,  $b$ , and  $c$  is not isolated. This makes the type of  $x$  over  $a$  and  $b$  an example of an *eventually non-isolated type*: over sufficiently many realizations of itself, it becomes non-isolated.

The important point here is that we can add a new point to  $F_{ab}$  without forcing any new points to appear either in any other fiber or in  $A$ . (Indeed, we can continue adding points to various fibers without ever forcing any unintended points to appear in other fibers or in  $A$ .) This is what is meant by saying that the types of generic elements of distinct fibers are *orthogonal*.

We use dimensions to code an undirected graph  $G$  on  $A$  into a model of this theory  $T$ . (Here the *dimension* of  $F_{ab}$  is just the number of  $\mathbb{Z}$ -chains in  $F_{ab}$ .) Starting with the prime model of  $T$ , we add one new element (hence a new  $\mathbb{Z}$ -chain) to each fiber  $F_{ab}$  for which the graph has an edge between  $a$  and  $b$ . The orthogonality ensures the accuracy of this coding, by guaranteeing that this process does not accidentally give rise to new elements in any fiber  $F_{ab}$  for which the graph had no edge between  $a$  and  $b$ . This builds a new model  $\mathfrak{M}$  of  $T$ , and the permutations of  $A$  which extend to automorphisms of  $\mathfrak{M}$  are exactly the automorphisms of  $G$ .

It now follows that there exist continuum-many countable pairwise non-isomorphic models of  $T$ , since an isomorphism  $f$  between two such structures

$\mathfrak{A}$  and  $\mathfrak{B}$  would have to map the set of indiscernibles in  $\mathfrak{A}$  onto that in  $\mathfrak{B}$ , hence likewise for the fibers, and therefore  $f$  on the indiscernibles would define an isomorphism between the graphs coded into  $\mathfrak{A}$  and  $\mathfrak{B}$ . Moreover, the graph  $G$  coded into  $\mathfrak{A}$  can be recovered from the computable infinitary  $\Sigma_2$ -theory of  $\mathfrak{A}$  – that is, we can compute a copy of  $G$  if we know this theory – and in fact we can enumerate the edges in a copy of  $G$  just from the computable infinitary  $\Pi_1$ -theory of  $\mathfrak{A}$ , since this much information allows us to recognize any two elements of  $F_{ab}$  in  $\mathfrak{A}$  that realize the nonisolated 2-type.

We will use this same strategy to code graphs into countable models  $K$  of  $\mathbf{DCF}_0$ , using the set  $A$  of indiscernibles given by Lemma 1.6. The fiber  $F_{mn}$  for  $a_m, a_n \in A$  will be the Manin kernel  $E_{a_m a_n}^\#$  defined in Theorem 1.2, which is shown in Theorem 1.3 to have the appropriate properties, and the non-isolated computable infinitary  $\Pi_1$ -type in  $F_{mn}$  will be the type of an element of  $F_{mn}$  whose coordinates are both transcendental over  $\mathbb{Q}\langle a_m + a_n \rangle$ . With this background, the reader should be ready to proceed with Theorem 3.1.

Although we will not attempt to generalize here, it is reasonable to guess that the procedure in Section 3 should work for other classes of countable structures for which the same conditions hold. Indeed, if the conditions hold for types using computable infinitary  $\Pi_n$ -formulas, then we conjecture that the same procedure allows one to code a graph  $G$  into a structure  $\mathfrak{A}$  in  $\mathcal{C}$  in such a way that the computable infinitary  $\Pi_n$ -theory of  $\mathfrak{A}$  allows one to enumerate the edges in a copy of  $G$ . In this case, the spectrum of  $\mathfrak{A}$  ought to contain exactly those Turing degrees  $\mathbf{d}$  whose  $n$ -th jump  $\mathbf{d}^{(n)}$  can enumerate the edges in a copy of  $G$ . That is,  $\text{Spec}(\mathfrak{A})$  should be the preimage of  $\text{Spec}(H)$  under the  $n$ -th jump operator (for the  $H$  defined from  $G$  in Lemma 3.2 below). On the other hand, there is no obvious reason why Theorem 4.1 need hold for countable models of such a theory.  $\mathbf{DCF}_0$  may be unusual in possessing both ENI-DOP (witnessed by  $\Pi_1$ -computable formulas) and the property that all of its  $\text{low}_1$  models are computably presentable.

### 3 Noncomputable Differentially Closed Fields

In this section we consider countable models of the theory  $\mathbf{DCF}_0$  which have no computable presentations. Using countable graphs with known spectra, we show how to construct differentially closed fields with spectra derived from those of the graphs. In particular, we create numerous countable differentially

closed fields which are not computably presentable (that is, whose spectra do not contain the degree  $\mathbf{0}$ ). We show that models of  $\mathbf{DCF}_0$  can have proper  $\alpha$ -th jump degree for every computable nonzero ordinal  $\alpha$ . However, we will see in Section 5 that this is impossible when  $\alpha = 0$ : no countable model of  $\mathbf{DCF}_0$  can have a least degree in its spectrum, unless that degree is  $\mathbf{0}$ . We encourage the reader to review Section 2 in order to understand the framework for the proof of the following theorem.

**Theorem 3.1** *Let  $G$  be a countable symmetric irreflexive graph. Then there exists a countable differentially closed field  $\widehat{K}$  of characteristic 0 such that*

$$\text{Spec}(\widehat{K}) = \{\mathbf{d} : \mathbf{d}' \text{ can enumerate a copy of } G\}.$$

(Saying that a degree  $\mathbf{c}$  can enumerate a copy of  $G$  means that there is a graph on  $\omega$ , isomorphic to  $G$ , whose edge relation is  $\mathbf{c}$ -computably enumerable.)

*Proof.* Taking  $G$  to have domain  $\omega$ , we first describe one presentation of  $\widehat{K}$ , on the domain  $\omega$ , without regard to effectiveness. We begin with a differential field  $\widehat{\mathbb{Q}}$  isomorphic to the differential closure of the constant field  $\mathbb{Q}$ . Recall from Subsection 1.2 that the following is a computable infinite set of indiscernibles:

$$A = \{y \in \widehat{\mathbb{Q}} : y' = y^3 - y^2 \text{ \& } y \neq 0 \text{ \& } y \neq 1\}.$$

We list the elements of  $A$  as  $a_0, a_1, \dots$ , and use  $a_n$  as our representative of the node  $n$  from  $G$ .

For each  $a_m$  and  $a_n$  with  $m < n$ , let  $E_{a_m a_n}$  be the elliptic curve defined by the equation  $y^2 = x(x - 1)(x - a_m - a_n)$ . The type of a differential transcendental is orthogonal to each strongly minimal set defined over  $\widehat{\mathbb{Q}}$ . Thus, for each  $m < n$ , the Manin kernel  $E_{a_m a_n}^\sharp$  contains only points algebraic over  $\mathbb{Q}\langle a \rangle$ . These sets are also orthogonal to  $A$ . The points of  $E_{a_m a_n}$  in  $(\widehat{\mathbb{Q}})^2$  form an abelian group, with (for each  $k > 0$ ) exactly  $k^2$  points whose torsion divides  $k$ , and with no non-torsion points, since  $\widehat{\mathbb{Q}}$  is the prime model of the theory  $\mathbf{DCF}_0$  over  $\mathbb{Q}$ . We will code our graph using these Manin kernels  $E_{a_m a_n}^\sharp$ , by adding a new point (with coordinates transcendental over  $\mathbb{Q}\langle a_m + a_n \rangle$ ) to our differential field just if the graph contains an edge from  $m$  to  $n$ . Any two of these Manin kernels are orthogonal, so adding a point to one (or to finitely many) of them will not add points to any other. Similarly, adding points to the Manin kernels will not add new points to  $A$ .

Now we build a differential field extension  $K$  of  $\widehat{\mathbb{Q}}$ , by adjoining to  $\widehat{\mathbb{Q}}$  exactly one new point  $x_{mn}$  of  $E_{a_m a_n}^\sharp$  for each  $m < n$  such that  $G$  has an edge between its nodes  $m$  and  $n$ . (We note that the type of such a generic point of  $E_a^\sharp$  over  $\widehat{\mathbb{Q}}$  is given by saying that  $x_{mn}$  is in  $E_a^\sharp$  but is not algebraic over  $\mathbb{Q}\langle a_m + a_n \rangle$ , hence is a computable type.) Adjoining all these  $x_{mn}$  yields a differential field  $K$ , and the differential field we want is the differential closure  $\widehat{K}$  of this  $K$ . The principal relevant feature of  $\widehat{K}$  is that, because of the mutual orthogonality of the Manin kernels,  $E_{a_m a_n}^\sharp(\widehat{K})$  contains a point non-algebraic over  $\mathbb{Q}\langle a_m + a_n \rangle$  if and only if there is an edge between  $m$  and  $n$  in  $G$ .

Now we claim that the spectrum of this  $\widehat{K}$  contains exactly those Turing degrees whose jumps can enumerate a copy of  $G$ . To show that every degree in the spectrum has this property, suppose that  $L \cong \widehat{K}$  has degree  $\mathbf{d}$ . Then with a  $\mathbf{d}$ -oracle, we can decide the set of all nontrivial solutions  $b_0 < b_1 < \dots$  in  $L$  to  $y' = y^3 - y^2$ . (The trivial solutions are 0 and 1, which we can recognize as the unique solutions to  $y^2 = y$ .) We build a graph  $H$ , with domain  $\omega$ , using a  $\mathbf{d}'$ -oracle. The oracle tells us, for each  $m < n$  and each solution  $(x, y) \in E_{b_m b_n}(L)$ , whether or not  $x$  is algebraic over  $\mathbb{Q}\langle b_m + b_n \rangle$ . If so, then we go on to the next point in  $L(E_{b_m b_n})$ . If  $x$  is not algebraic, then we enumerate an edge between  $m$  and  $n$  into our graph  $H$ . The graph  $H$  thus enumerated is isomorphic to  $G$ : the isomorphism  $f$  from  $L$  onto  $\widehat{K}$  must map the set  $\{b_0, b_1, \dots\}$  bijectively onto the set  $\{a_0, a_1, \dots\}$ , and the map sending each  $m \in H$  to the unique  $n \in G$  such that  $f(b_m) = a_n$  will be an isomorphism of graphs. Thus  $\mathbf{d}'$  has enumerated a copy  $H$  of  $G$ , as required.

Conversely, suppose that the Turing degree  $\mathbf{d}'$  can enumerate a graph  $H$  isomorphic to  $G$ . Specifically, for a fixed set  $D \in \mathbf{d}$ , there is a Turing functional  $\Phi$  for which the partial function  $\Phi^{D'}$  has domain  $\{\langle m, n \rangle : H \text{ has an edge from } m \text{ to } n\}$ . The procedure above essentially describes how to build a differentially closed field  $\widehat{L}$  below a  $\mathbf{d}$ -oracle with  $\widehat{L} \cong \widehat{K}$ . Using Theorem 1.1, start building a computable copy of  $\widehat{\mathbb{Q}}$ , in which we enumerate all nontrivial solutions  $b_n$  to  $y' = y^3 - y^2$ , but build this solution slowly, with one new element at each stage, so that each step  $L_s$  in this construction is actually a finite fragment of the differential field  $L$  we wish to build. Then, with the  $\mathbf{d}$ -oracle, enumerate the jump  $D'$  of the set  $D \in \mathbf{d}$ : say  $D' = \cup_{s \in \omega} D'_s$ . Whenever we find a stage  $s$  such that some  $\langle m, n \rangle$  lies in  $\text{dom}(\Phi_s^{D'_s})$  (and did not lie in this domain for  $s-1$ ), we adjoin to  $L_s$  a new point  $(x_{m,n,s}, y_{m,n,s})$  in  $E_{b_m b_n}^\sharp$ , such that  $x_{m,n,s}$  does not yet satisfy any nonzero differential poly-

mial at all over  $L_s$ , and is specified not to be a zero of the first  $s$  polynomials of degree  $\leq s$  over  $L_s$ . Of course,  $y_{m,n,s}$  is a zero of the curve  $E_{b_m b_n}$  over  $x_{m,n,s}$ ; this fully determines  $y_{m,n,s}$  and its derivatives in terms of  $L_s$  and  $x_{m,n,s}$  and its derivatives.

At the next stage, if we still have  $\langle m, n \rangle \in \text{dom}(\Phi_{s+1}^{D'_{s+1}})$ , then we declare that  $x_{m,n,s+1} = x_{m,n,s}$  is not a zero of any of the first  $s+1$  polynomials of degree  $\leq s+1$  over  $L_{s+1}$ , and so on for subsequent stages. If we ever reach a stage  $t > s$  at which  $\langle m, n \rangle \notin \text{dom}(\Phi_t^{D'_t})$  (which is possible, if the oracle has changed from the previous stage), then we turn  $(x_{m,n,s}, y_{m,n,s})$  into a  $k$ -torsion point, with  $k \geq t$  being the smallest value for which this is consistent with the finite fragment  $L_{t-1}$  built up till then. Since non-torsion points realize non-principal types, the finitely many facts we have enumerated so far about  $L_{t-1}$  cannot possibly force this point to be a non-torsion point, so for some  $k$  this will be possible, and by searching we can identify such a  $k$ , using the decidability of the complete theory  $\mathbf{DCF}_0$ . As we subsequently continue to build  $L$  (including the cofinite portion of  $\widehat{\mathbb{Q}}$  which is yet to be constructed), we will take this  $k$ -torsion point into account, treating it as part of  $\widehat{\mathbb{Q}}$ . The decidability of  $\mathbf{DCF}_0$  makes it easy to include the point into  $\widehat{\mathbb{Q}}$  and still know what to build at each subsequent step.

Thus the existence of a nonalgebraic point on  $E_{b_m b_n}^\sharp$  in the field  $L$  built by this process is equivalent to  $\langle m, n \rangle$  actually lying in  $\text{dom}(\Phi^{D'})$ , and for all  $\langle m, n \rangle$  not in this domain, every pair  $(x_{m,n,s}, y_{m,n,s})$  ever defined (for any  $s$ ) was eventually turned into a torsion point, meaning that it wound up in the subfield  $\widehat{\mathbb{Q}}$  of  $L$ , since this subfield contains all  $k^2$  of the  $k$ -torsion points for  $E_{b_m b_n}$  in  $L$ . Therefore, the  $L$  that we finally built is just the differential field extension of  $\widehat{\mathbb{Q}}$  by one nontorsion point for each edge in  $H$ , and the differential closure  $\widehat{L}$  of this  $L$  is isomorphic to  $\widehat{K}$ , and is also  $\mathbf{d}$ -computable, by Theorem 1.1. This completes the proof of the theorem. ■

Next we show that in Theorem 3.1, it is reasonable to replace the graph  $G$ , which the  $\mathbf{d}'$ -oracle can enumerate, by another countable graph  $H$  which the same oracle can actually compute.

**Lemma 3.2** *Let  $G$  be a countable (symmetric irreflexive) graph. Then there exists a countable graph  $H$  such that*

$$\text{Spec}(G) = \{\mathbf{d} : \mathbf{d} \text{ can enumerate a copy of } H\}.$$

*Conversely, for every countable graph  $H$ , there exists a countable graph  $G$  whose spectrum contains exactly those Turing degrees which can enumerate a copy of  $H$ .*

*Proof.* This is simply a question of coding. Fix  $G$ , with domain  $\omega$ . For each node  $n$  in  $G$ , we create five nodes in  $H$ : a node  $x_n$  which is the actual node coding  $n$ , and four other nodes which form a copy of the complete graph  $K_4$ . We place an edge between  $x_n$  and exactly one of the four nodes in this  $K_4$ , and the nodes in the  $K_4$  will not be adjacent to any other nodes in  $H$ . The copy of  $K_4$  may be seen as a “tag” for  $x_n$ , identifying it as a coding node.

Next, having created a coding node  $x_n$  for each  $n$ , we consider each pair  $m < n$ . If there is an edge between  $m$  and  $n$  in  $G$ , then we place a path of length 6 from  $x_m$  to  $x_n$  in  $H$ , adding five nodes, each adjacent to the next, plus edges from  $x_m$  to the first and from the last to  $x_n$ . If there is no edge between  $x_m$  and  $x_n$  in  $G$ , then instead we place a path of length 9 from  $x_m$  to  $x_n$  in  $H$ , by adding eight new nodes. In both cases, the nodes along the path (except  $x_m$  and  $x_n$ ) have valence 2 in  $H$ : they are not adjacent to anything in  $H$  except the preceding and succeeding nodes on their path. This completes the construction of  $H$ . It is clear that this particular  $H$  is computable in  $G$ , and more generally that the same process with any  $\tilde{G} \cong G$  would enumerate an  $\tilde{H} \cong H$ , so that  $\text{Spec}(G)$  contains only degrees which can enumerate copies of  $H$ .

On the other hand, suppose that a degree  $\mathbf{d}$  can enumerate the edge relation in a graph  $\tilde{H} \cong H$ . We give a construction of a  $\mathbf{d}$ -computable graph  $\tilde{G} \cong G$ , with domain  $\omega$ . With a  $\mathbf{d}$ -oracle, start enumerating  $H$ . Whenever we see a copy of  $K_4$  enumerated, wait for one of its four nodes to become adjacent to some fifth node; then label that fifth node  $x_n$  (starting with  $x_0$  for the first copy of  $K_4$ , then  $x_1$  for the next one we find, and so on). To decide whether there is an edge between  $m$  and  $n$  in  $\tilde{G}$ , wait until this process has named nodes  $x_m$  and  $x_n$  in  $\tilde{H}$ , and then wait until the  $\mathbf{d}$ -oracle enumerates edges forming a path of length either 6 or 9 from  $x_m$  to  $x_n$ . If this path has length 6, then  $m$  and  $n$  are adjacent in  $\tilde{G}$ , while, if the path has length 9, they are not. To see that this  $\tilde{G}$  must be isomorphic to  $G$ , one simply checks that there are no copies of  $K_4$  in  $H$  except the ones which we added as tags for nodes  $x_n$ , and that all paths from any  $x_m$  to any  $x_n$  with  $m < n$  which go through any other node  $x_p$  must have length  $\geq 12$ . All this is clear from the construction of  $H$ , and so  $\tilde{G}$  is indeed a  $\mathbf{d}$ -computable copy of  $G$ . Thus every degree which can enumerate a copy of  $H$  lies in the spectrum of  $G$ .

Next we turn to the converse, starting with a graph  $H$  (and a degree  $\mathbf{c}$  which can enumerate  $H$ ) and building a corresponding  $\mathbf{c}$ -computable  $G$ . Now each node  $n \in H$  has a representative  $y_n$  in  $G$ , and again each  $y_n$  is adjacent to a single node within a copy of  $K_4$ , with the four nodes in the  $K_4$  adjacent to nothing else except each other. All these nodes (countably many copies of  $K_4$ , each with one  $y_n$  attached to it) constitute  $G_0$ . At each subsequent stage  $s + 1$ , if our copy of  $H$  enumerates an edge between some nodes  $m$  and  $n$  in  $H$ , we add a new node to  $G_{s+1}$  and make it adjacent to both  $y_m$  and  $y_n$  (but not adjacent to anything else). This is the entire construction of  $G$ , and  $G$  is computable in the degree  $\mathbf{c}$  which enumerated our copy of  $H$ , because with that oracle, whenever a new node was added to  $G$ , we decided immediately which of the already-existing nodes in  $G$  were adjacent to it. Moreover, it is clear that, whenever any degree  $\mathbf{d}$  can enumerate a graph  $\tilde{H} \cong H$ , this same process will build a  $\mathbf{d}$ -computable graph  $\tilde{G} \cong G$ . For the reverse inclusion, suppose that  $\mathbf{d}$  computes a graph  $\tilde{G} \cong G$ . Whenever a copy of  $K_4$  appears in this  $\tilde{G}$ , we watch for the unique fifth node adjacent to one element of the  $K_4$  to appear, and when it does, we call it  $\tilde{y}_n$  (for the least  $n$  such that we have not already defined  $\tilde{y}_n$  in  $\tilde{G}$ ) and add a new node  $n$  to our  $\tilde{H}$  for  $\tilde{y}_n$  to represent. Then, when and if we discover a node in  $\tilde{G}$  adjacent to both  $\tilde{y}_m$  and  $\tilde{y}_n$ , we enumerate an edge between  $m$  and  $n$  in our  $\tilde{H}$ . Thus we have enumerated an  $\tilde{H}$  isomorphic to  $H$ , computably in the degree  $\mathbf{d}$  of the copy  $\tilde{G}$  of  $G$ , completing the proof.  $\blacksquare$

Recall that, for a computable ordinal  $\alpha$ , the  $\alpha$ -th *jump degree* of a countable structure  $\mathcal{S}$  is the least degree in the set  $\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \text{Spec}(\mathcal{S})\}$ .

**Corollary 3.3** *For every graph  $H$ , there exists a differentially closed field  $K$  such that*

$$\text{Spec}(K) = \{\mathbf{d} : \mathbf{d}' \in \text{Spec}(H)\}.$$

*In particular, for every computable ordinal  $\alpha > 0$  and every degree  $\mathbf{c} >_T \mathbf{0}^{(\alpha)}$ , there is a differentially closed field which has  $\alpha$ -th jump degree  $\mathbf{c}$ , but has no  $\gamma$ -th jump degree whenever  $\gamma < \alpha$ .*

Using ordinal addition, one can re-express the second result by stating that, for every  $\beta < \omega_1^{CK}$  and every  $\mathbf{c}$  with  $\mathbf{c} >_T \mathbf{0}^{(1+\beta)}$ , there is a differentially closed field  $K$  with proper  $(1 + \beta)$ -th jump degree  $\mathbf{c}$ .



*Proof.* Given  $H$ , use Lemma 3.2 to get a graph  $G$  whose copies are enumerable by precisely the Turing degrees in  $\text{Spec}(H)$ . Then apply Theorem 3.1 to this  $G$  to get the differentially closed field  $K$  required, with

$$\text{Spec}(K) = \{\mathbf{d} : \mathbf{d}' \text{ can enumerate a copy of } G\} = \{\mathbf{d} : \mathbf{d}' \in \text{Spec}(H)\}.$$

Now, for every computable ordinal  $\beta$  and every degree  $\mathbf{c} \geq \mathbf{0}^{(\beta)}$ , there exists a graph  $H$  with  $\beta$ -th jump degree  $\mathbf{c}$ , but with no  $\gamma$ -th jump degree for any  $\gamma < \beta$ . (This is shown for linear orders in [1] and [5] for all  $\beta \geq 2$ , and Theorem 1.9 then transfers the result to graphs. For  $\beta < 2$  it is a standard fact; see e.g. [6].) If  $\alpha > 0$  is finite, let  $\beta$  be its predecessor and apply the first part of the corollary to the  $H$  corresponding to  $\mathbf{c}$  and to this  $\beta$ . Then

$$\{\mathbf{d}^{(\beta)} : \mathbf{d} \in \text{Spec}(H)\} = \{(\mathbf{d}')^{(\beta)} : \mathbf{d} \in \text{Spec}(K)\} = \{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \text{Spec}(K)\},$$

so  $\mathbf{c}$  is the  $\alpha$ -th jump degree of  $K$ . On the other hand, when  $\beta \geq \omega$ , the degree  $(\mathbf{d}')^{(\beta)}$  is just  $\mathbf{d}^{(\beta)}$  itself, and so, for every infinite computable ordinal  $\alpha$ , the above analysis with  $\beta = \alpha$  shows that again  $K$  has  $\alpha$ -th jump degree  $\mathbf{c}$ . In both cases, this also proves that for each  $\gamma < \alpha$ ,  $K$  has no  $\gamma$ -th jump degree. ■

## 4 Low Differentially Closed Fields

Corollary 3.3 demonstrated that, for every nonlow Turing degree  $\mathbf{d}$ , there exists a  $\mathbf{d}$ -computable differentially closed field with no computable presentation: with  $\mathbf{d}' > \mathbf{0}'$ , just take the model of  $\mathbf{DCF}_0$  given by the corollary with jump degree  $\mathbf{d}'$ . (The corollary showed not only that this structure has jump degree  $\mathbf{d}'$ , but also that every degree whose jump computes  $\mathbf{d}'$  lies in its spectrum. In particular, the structure has a  $\mathbf{d}$ -computable copy.) Of course, there do exist noncomputable low Turing degrees  $\mathbf{d}$ , that is, degrees with  $\mathbf{d} > \mathbf{0}$  but  $\mathbf{d}' = \mathbf{0}'$ . Corollary 3.3 does not yield any method for proving the same result for these degrees. Indeed, the surprising answer, to be proven in this section, is that when  $\mathbf{d}$  is low, every  $\mathbf{d}$ -computable differentially closed field has the degree  $\mathbf{0}$  in its spectrum.

**Theorem 4.1** *Every low differentially closed field  $K$  of characteristic 0 is isomorphic to a computable differential field.*

*Proof.* The goal of our construction is to build a computable differential field  $F$ , with domain  $\{y_0, y_1, \dots\}$ , and a sequence of uniformly computable finite partial functions  $h_s : \omega \rightarrow \omega$  such that, for every  $n$ ,  $h(n) = \lim_s h_s(n)$  converges to an element  $m \in F$  and thus defines an isomorphism  $x_n \mapsto y_{h(n)}$  from  $K$  onto  $F$ . When  $n \leq h(n)$ , we will arrange that the minimal differential polynomial of  $x_n$  over the differential subfield generated by the higher-priority elements of  $K$ :

$$\mathbb{Q}\langle x_0, x_{h^{-1}(0)}, x_1, x_{h^{-1}(1)}, \dots, x_{n-1}, x_{h^{-1}(n-1)} \rangle \subseteq K$$

equals the minimal differential polynomial of  $y_{h(n)}$  over the corresponding differential subfield

$$\mathbb{Q}\langle y_{h(0)}, y_0, y_{h(1)}, y_1, \dots, y_{h(n-1)}, y_{n-1} \rangle \subseteq F.$$

More precisely, there will be a  $p_n \in \mathbb{Q}\{X_0, Y_0, X_1, \dots, Y_{n-1}, X_n\}$  such that  $p_n(x_0, x_{h^{-1}(0)}, x_1, x_{h^{-1}(1)}, \dots, x_{h^{-1}(n-1)}, X_n)$  is the minimal differential polynomial of  $x_n$  over the first subfield and  $p_n(y_{h(0)}, y_0, y_{h(1)}, y_1, \dots, y_{n-1}, Y_n)$  is the minimal differential polynomial of  $y_{h(n)}$  over the second subfield.

Likewise, when  $n > h(n)$ , we will arrange that the minimal differential polynomial of  $x_n$  over the differential subfield

$$\mathbb{Q}\langle x_0, x_{h^{-1}(0)}, x_1, x_{h^{-1}(1)}, \dots, x_{h^{-1}(h(n)-1)}, x_{h(n)} \rangle \subseteq K$$

is equal to the minimal differential polynomial of  $y_{h(n)}$  over

$$\mathbb{Q}\langle y_{h(0)}, y_0, y_{h(1)}, y_1, \dots, y_{h(n)-1}, y_{h(h(n))} \rangle \subseteq F.$$

(With  $n > h(n)$ , the lower index  $h(n)$  gives the priority of the pair  $(x_n, y_{h(n)})$ . Those pairs containing any of the elements  $x_0, \dots, x_{h(n)}$  and  $y_0, \dots, y_{h(n)-1}$  will have higher priority and so will be considered first.)

This will establish that  $h$  defines an embedding of differential fields. Moreover, we will also ensure that  $h$  is a bijection from  $\omega$  onto  $\omega$ , so that it actually defines an isomorphism. Since  $F$  is computable, this will prove the theorem. Notice that, in contrast to the situation with Boolean algebras, it will follow that every low differentially closed field is  $\Delta_2^0$ -isomorphic to a computable one; for Boolean algebras a  $\Delta_3^0$ -isomorphism is sometimes required.

Clearly, executing this construction will require us to figure out minimal differential polynomials of various elements of  $K$  over various subfields. The given differential field  $K$ , being low, has all its functions computable in some

Turing degree  $\mathbf{d}$  for which  $\mathbf{d}' = \mathbf{0}'$ . It follows, first, that these functions are all computably approximable, and moreover, that there is a computable function which converges to the characteristic function of the  $\Sigma_1$ -fragment of the elementary diagram of  $K$ . In fact, even more is true: this computable function approximates the truth of computable infinitary  $\Sigma_1$  formulas on elements of  $K$ , since  $\mathbf{d}'$  is sufficient to determine the truth of such formulas. For example, the following statement about the first  $k+1$  elements  $x_0, \dots, x_k$  in the domain of  $K$ :

$$(\exists p \in \mathbb{Q}\{X_0, \dots, X_n\})[p(x_0, \dots, x_n) = 0 \ \& \ \text{ord}_{X_n}(p) \geq 0]$$

is arithmetically  $\Sigma_1$  over the atomic diagram of  $K$ , and therefore  $\mathbf{d}'$ -decidable, even though it quantifies over arbitrarily long finite tuples from  $K$  (namely, the tuples of coefficients for polynomials in  $\mathbb{Q}\{X_0, \dots, X_n\}$ ) and thus is not a finitary formula. This statement says that  $x_n$  is differentially algebraic over the differential subfield  $K_{n-1} = \mathbb{Q}\langle x_0, \dots, x_{n-1} \rangle$  of  $K$ . (Recall that  $\text{ord}_{X_n}(p)$  is the greatest  $r$  for which the  $r$ -th derivative  $X_n^{(r)}$  appears in  $p$ ; if  $p$  does not involve  $X_n$  at all, then this order is  $-1$ .) Therefore, we have a computable predicate  $\text{Trans}_s$  such that, for every  $n$  and every finite ordered tuple  $\rho \in K^{<\omega}$ ,  $\text{Trans}(x_n, \rho) = \lim_s \text{Trans}_s(x_n, \rho)$  converges and

$$\text{Trans}(x_n, \rho) = \begin{cases} 1, & \text{if } x_n \text{ is differentially transcendental over } \mathbb{Q}\langle \rho \rangle; \\ 0, & \text{if } x_n \text{ is differentially algebraic over } \mathbb{Q}\langle \rho \rangle. \end{cases}$$

Similarly, we have a computable function  $M(n, \rho, p, s)$ , uniform in  $s$ , in  $n$ , in  $\rho = (x_{n_1}, \dots, x_{n_k}) \in K^{<\omega}$ , and in  $p \in \mathbb{Q}\{X_1, \dots, X_k, X\}$ , whose limit as  $s \rightarrow \infty$  is 1 if  $p(x_{n_1}, \dots, x_{n_k}, X)$  is the minimal differential polynomial of  $x_n$  over  $\mathbb{Q}\langle \rho \rangle$  (that is, if  $p(x_{n_1}, \dots, x_{n_k}, x_n) = 0$  and  $p$  is monic and algebraically irreducible with  $\text{ord}_X(p) \geq 0$  and no other differential polynomial with these properties has lower rank, as defined in Section 1), and 0 otherwise. Of course,  $\text{Trans}(x_n, \rho) = 1$  if and only if this limit is 0 for every  $p$ , while if  $\text{Trans}(x_n, \rho) = 0$ , then there is exactly one  $p$  for which this limit is 1. Therefore, using a speed-up procedure as needed, we may define a computable function  $p_{n,\rho,s}$  which will converge to the minimal differential polynomial of  $x_n$  over  $\mathbb{Q}\langle \rho \rangle$ , under the convention that if  $x_n$  is differentially transcendental over this subfield, then its minimal differential polynomial is the zero polynomial.

$$p_{n,\rho,s} = \begin{cases} 0, & \text{if } \text{Trans}_s(x_n, \rho) = 1; \\ p(X_1, \dots, X_k, X), & \text{if } \text{Trans}_s(x_n, \rho) = 0 \text{ and } p \text{ has least rank} \\ & \text{in } X \text{ among all } q \text{ with } M(n, \rho, q, s) = 1. \end{cases}$$

**Notation 4.2** *To avoid cumbersome subscripts, we adopt the convention of writing “[ $s$ ]” at the end of an expression to indicate that all items in the expression have the values assigned to them as of stage  $s$ . For example,  $p_{n_i, \rho_i}(y_{h(n_0)}, \dots, y_{h(n_i)})[s]$  will denote  $p_{n_{i,s}, \rho_{i,s}}(y_{h_s(n_{0,s})}, \dots, y_{h_s(n_{i,s})})$ .*

It will simplify matters for us to treat  $F$  and  $K$  as relational structures, with addition and multiplication as three-place relations and differentiation as a two-place relation. Since our  $F$  will be isomorphic to  $K$  (as a relational structure), these relations will define functions in  $F$ , and since the relations will be computable in  $F$ , these functions will also be computable there. However, using relational structures will allow us not to have to close under the operations too promptly. For example, if at some stage  $s$  we are thinking of an element  $y \in F_s$  as a differential transcendental (over the ground field  $\mathbb{Q}$ , say), we do not automatically add all of its derivatives to  $F$  right away. If later on we find that we want it not to be transcendental after all, we can still place an algebraic relation on its derivatives, or even make some high derivative  $y^{(r)}$  equal 0 in  $F$ . Of course, the construction of  $F$  will eventually produce every derivative  $y^{(r)}$ , effectively, but we prefer not to add them all to  $F$  at once. In particular we will write  $K_s^0$  for the finite relational structure of  $K$  on the domain  $\{0, 1, x_0, x_1, \dots, x_s\}$  (where  $K$  has domain  $\{x_0, x_1, \dots\}$ ), noting that the structure  $K_s^0$  cannot be computed uniformly in  $s$ .  $K_s^0$  will denote the differential subfield of  $K$  generated by  $K_s^0$ . Likewise,  $F_s^0$  will also consist of finitely many elements, which together generate a differential subfield  $F_s^0$  of  $F = \cup_t F_t^0$ , and we will define the structure of  $F$  solely by a decision procedure determining which differential polynomials  $f \in \mathbb{Q}\{Y_0, \dots, Y_k\}$  (for every  $k$ ) satisfy  $f(y_0, \dots, y_k) = 0$  in  $F$ . We enumerate the set  $U = \cup_s U_s$  of those  $f(Y_0, \dots, Y_k)$  for which this holds, and ensure that, for every  $f$ , there is a unique  $m$  for which the polynomial  $(f - Y_m)$  lies in  $U$ . (Essentially this says that we are setting  $f(y_0, \dots, y_k) = y_m$  in  $F$ .) This condition will make  $U$  computable, since, for any  $f$ , we can enumerate  $U$  until we find an  $m$  with  $(f - Y_m) \in U$ ; then  $f \in U$  iff  $m = 0$  (with  $y_0$  being the zero element of  $F$ ). This will be sufficient information to compute all the operations in  $F$ .

In addition, in this relational language, having  $F_s^0$  be a finite fragment of a differential field will allow us to lean heavily on the theory  $\mathbf{DCF}_0$  for guidance in constructing  $F_{s+1}^0$ . This theory is complete and decidable, and so, given the finite fragment  $F_s^0$  containing (say)  $y_0, \dots, y_r$ , we can write out the entire relational atomic diagram  $\psi(y_0, \dots, y_r)$  of these elements. When

considering how to build  $F_{s+1}^0$ , we can then ask whether a sentence such as

$$\exists Y_0 \cdots \exists Y_m [\psi(Y_0, \dots, Y_r) \ \& \ q(Y_0, \dots, Y_m) = 0]$$

lies in  $\mathbf{DCF}_0$ . (Here  $q$  is some differential polynomial over  $\mathbb{Q}$  for which we might wish to declare  $\vec{y}$  to be a zero.) If this is inconsistent, then the decision procedure for  $\mathbf{DCF}_0$  will tell us so, and we will not set  $q(\vec{y}) = 0$  in  $F_{s+1}^0$ . If it is consistent (and hence belongs to the complete theory  $\mathbf{DCF}_0$ ), then some tuple of elements of  $K$  must realize  $[\psi(\vec{X}) \ \& \ q(\vec{X}) = 0]$ , and so it is safe for us to set  $q(\vec{y}) = 0$  in  $F_{s+1}^0$ .

The requirements for the construction concern the function  $h : \omega \rightarrow \omega$  which we build as the limit  $\lim_s h_s$  of a computable sequence of functions. In order to make  $h$  a bijection, we will satisfy for each  $m$  and  $n$ :

$$\mathcal{R}_n : h(n) = \lim_s h_s(x_n) \text{ exists.}$$

$$\mathcal{S}_m : h^{-1}(m) = \lim_s h_s^{-1}(y_m) \text{ exists.}$$

Additionally, to make  $h$  define an isomorphism of differential fields, we will build a sequence  $\langle n_i \rangle_{i \in \omega}$ , where  $i \mapsto n_i$  is a bijection from  $\omega$  onto itself, and ensure, for each  $i$  and each  $p \in \mathbb{Q}\{X_0, \dots, X_i\}$ :

$$p(x_{n_0}, \dots, x_{n_i}) = 0 \text{ in } K \iff p(y_{h(n_0)}, \dots, y_{h(n_i)}) = 0 \text{ in } F.$$

It will then follow, by induction on  $i$ , that each  $x_{n_i}$  satisfies the same 1-type over  $K_{i-1} = \mathbb{Q}\langle x_{n_0}, \dots, x_{n_{i-1}} \rangle$  that  $y_{h(n_i)}$  satisfies over  $\mathbb{Q}\langle y_{h(n_0)}, \dots, y_{h(n_{i-1})} \rangle$ , so that  $h$  will define an isomorphism. These requirements are given a priority ranking, with  $\mathcal{R}_i \prec \mathcal{S}_i \prec \mathcal{R}_{i+1}$  for all  $i$  (meaning that among these three,  $\mathcal{R}_i$  has the highest priority, then  $\mathcal{S}_i$ , then  $\mathcal{R}_{i+1}$ ).

At stage 0, we set  $F_0^0$  to contain  $y_0 = 0$  and  $y_1 = 1$  as the identity elements of  $F$ . We also use Theorem 1.1 to define a Rabin embedding of  $\mathbb{Q}$  into a computable presentation  $\widehat{F}_0 = \widehat{F}_{0,0} = \widehat{F}_{1,0}$  of its differential closure  $\widehat{\mathbb{Q}}$ , for reasons explained below. We set  $g_0(y_0)$  and  $g_0(y_1)$  equal to the identity elements in  $\widehat{F}_0$ . It would be natural to define triples such as  $(y_0, y_0, y_0)$  and  $(y_0, y_1, y_1)$  to lie in the addition relation for  $F_0$ , and  $(y_0, y_0)$  and  $(y_1, y_0)$  to lie in the differentiation relation, and so on. There would be no harm in doing so for finitely many tuples, but, in line with the construction of the rest of  $F$ , the actual step is that we add  $Y_0$  and  $(Y_1 - 1)$  to the set  $U_0$ , i.e., to the computable enumeration of the set  $U$  of those differential polynomials  $f \in \mathbb{Q}\{Y_0, Y_1, \dots\}$

for which  $f(y_0, y_1, \dots) = 0$  in  $F$ . This is equally strong, and as with the relations, it does not yet commit  $y_1 + y_1$  to equal any particular element of  $F$ . It also parallels our process for approximating  $K$ , which uses minimal differential polynomials rather than using the relations directly, and this will simplify the construction. In order to use the differential polynomials this way, though, we will need to be able to consider the finite set  $U_s$  at each stage and decide, for each  $m$ , just what minimal differential polynomial (over the higher-priority elements of  $F$ ) we have committed  $y_m$  to satisfy. The content of the following lemma is that we can do so.

**Lemma 4.3** *There is an algorithm which, when given as input a (strong index for a) finite set  $V \subseteq \mathbb{Q}\{T_0, \dots, T_r\}$  of differential polynomials and an  $m \leq r$  such that the sentence  $\exists T_0 \dots \exists T_r \psi$  lies in  $\mathbf{DCF}_0$ , where  $\psi$  is the formula*

$$\bigwedge_{g \in V} g(T_0, \dots, T_{k_g}) = 0 \ \& \ \bigwedge_{i < j \leq r} T_i \neq T_j,$$

*outputs the (unique) differential polynomial  $f = \sum_{\theta} f_{\theta} T_m^{\theta}$  in  $\mathbb{Q}\{T_0, \dots, T_m\}$  of least rank in  $T_m$  (written here using finitely many  $f_{\theta} \in \mathbb{Q}\{T_0, \dots, T_{m-1}\}$ ) such that the sentence*

$$(\forall T_0, \dots, \forall T_r [\psi \rightarrow f = 0]) \ \& \ \left( \exists T_0, \dots, \exists T_r \bigvee_{\theta} [\psi \ \& \ f_{\theta} \neq 0] \right)$$

*lies in  $\mathbf{DCF}_0$ . (Here again we consider the zero polynomial to have rank  $+\infty$ ; this will be the output of the algorithm if and only if the input is consistent with  $T_m$  being differentially transcendental over  $\mathbb{Q}\langle T_0, \dots, T_{m-1} \rangle$ .)*

(The point here is that committing ourselves to the finite set  $\psi$  of conditions will force  $T_m$  to be a zero of  $f$ , but will not force it to be a zero of any differential polynomial of lesser rank. So the algorithm is producing the apparent minimal differential polynomial  $f$  of  $T_m$  over  $T_0, \dots, T_{m-1}$ , under the condition  $\psi$ , although of course  $\psi$  does not necessarily rule out the possibility of  $T_m$  satisfying some differential polynomial of smaller rank as well.)

*Proof.* For the given  $V$ , the algorithm proceeds by recursion on  $m \leq r$ . For  $m = 0$ , we set  $L_0 = \mathbb{Q}$  and check whether  $V$  contains any differential polynomials in  $L_0\{T_0\}$ . If not, then the output  $f_0$  for  $m = 0$  is the zero polynomial, since it is consistent with  $\psi$  for  $T_0$  to be differentially transcendental over

$L_0$  (and indeed no other  $f$  would satisfy the conditions required). If  $V$  does contain polynomials in  $L_0\{T_0\}$ , then we use Ritt's algorithm to reduce them to a single differential polynomial of least possible rank. This algorithm, given in [23], is analogous to the reduction procedure for finding a principal generator of an algebraic ideal in the (non-differential) polynomial ring  $L[T]$ , where by multiplying and subtracting, one cancels the leading term of a polynomial whenever possible. Ritt's algorithm, given inputs  $g$  and  $h$  in  $L\{T\}$  (for any computable differential field  $L$ ; in this case with  $L = K_0$ ), produces the unique monic differential polynomial  $f_{gh}$  of least rank in  $L\{T\}$  such that, in all differential field extensions of  $L$ , the condition  $h(t) = g(t) = 0$  forces  $f_{gh}(t) = 0$  as well. (Quite possibly  $f_{gh} = g$  or  $f_{gh} = h$ , whichever has lower rank, but sometimes  $f_{gh}$  will be new, of strictly lower rank than both  $g$  and  $h$ .) We apply this to each pair of polynomials in  $V_0 = (V \cap L_0\{T_0\})$ , and to the polynomials  $f_{gh}$  produced in this way, until we have found an output  $f_0$  which, when reduced via Ritt's algorithm against every element in the closure of  $V_0$  under the operation  $(g, h) \mapsto f_{gh}$ , always returns  $f_0$  again. Such an  $f_0$  must exist, since ranks are ordinals, and this  $f_0$  is exactly the polynomial required by the lemma, in the case  $m = 0$ .

Having found  $f_0$ , we know that  $V$  forces  $f_0(T_0) = 0$ , and so we may treat  $T_0$  as a zero of  $f_0$ , taking  $L_1$  to be the fraction field of the differential ring  $L_0\{T_0\}/[f_0] \cdot h_{f_0}^\infty$ . (If  $f_0$  is the zero polynomial, this  $L_1$  is just the differential field  $L_0\langle T_0 \rangle$  generated by a single differential transcendental  $T_0$ . If not, then  $L_1$  is the differential field generated by an element whose minimal differential polynomial over  $L_0$  is  $f_0$ .) We let  $t_0$  be the image of  $T_0$  in  $L_1$ , change all polynomials  $g \in V$  to  $g(t_0, T_1, \dots, T_r) \in L_1\{T_1, \dots, T_r\}$ , and repeat the previous step for these polynomials over the computable differential field  $L_1$ , yielding the differential polynomial  $f_1(t_0, T_1)$  of least rank which is forced to equal 0 by the formula  $\psi$ . For  $m = 1$ , our algorithm therefore outputs  $f_1(T_0, T_1)$ . (Notice that for certain  $g(T_0, T_1) \in V$ ,  $g(t_0, T_1)$  might be the zero polynomial. If every  $g \in V \cap \mathbb{Q}\{T_0, T_1\}$  has this property, then  $f_1$  is defined to be the zero polynomial, i.e.,  $t_1$  is made differentially transcendental over  $L_1$ .)

The recursive process is now clear: to find  $f_{m+1}$ , we run the same process over  $L_{m+1}$ , which is defined from  $L_m$  using  $f_m$  just as  $L_1$  was defined from  $L_0 = \mathbb{Q}$  using  $f_0$ . This is the algorithm required by the lemma.  $\blacksquare$

The rest of the construction of  $F$  will take place over the  $F_0^0$  defined at stage 0, with some more explanation now before we proceed. The domain

of  $F$  will be  $\{y_m : m \in \omega\}$ , with all those  $y_m$  not in  $F_0^0$  adjoined to  $F$  at later stages. It is important that we do *not* define all of the algebraic closure  $\overline{F_0}$  within  $F$  at this point, and we explain here why. If an element  $x_n$  of  $K$  appears at some stage  $s$  to have minimal differential polynomial  $p_{n,s}(X_0, \dots, X_n) = \delta X_n$ , then  $h_s$  will map  $n$  to some  $m$ , setting  $\delta y_m = 0$  (by adding the polynomial  $\delta Y_m$  to  $U$ ). Later on, the approximation to  $K$  may change its mind and make  $x_n$  either algebraic over  $\mathbb{Q}$ , or else not a zero of this polynomial  $\delta X_n$  at all, and indeed  $K$  might turn out to have no transcendental constants whatsoever. In that case, our  $y_m$  will become an element of  $\overline{F_0}$ ; since its derivative is already set to 0 in the computable differential field  $F$ , this is our only option. If  $F_0$  itself (or any subsequent  $F_s$ ) had already contained the entire algebraic closure of  $F_0$ , then this option would have been closed off to us, and in this case  $y_m$  could not have any possible preimage in  $K$ . The point is that, for every  $q(X_n) \in \mathbb{Q}[X_n]$ , the unconstrainable polynomial  $\delta X_n$  (of order 1) must have zeroes which are not zeroes of  $q$ , but are zeroes of some other polynomial of order 0. (Otherwise this  $q$  would be a constraint on  $\delta X_n$ , which is impossible.) So we make sure that at each stage  $t$ , there will still be infinitely many of these  $q$  for which we have not yet committed ourselves about whether  $q(y_m) = 0$  in  $F$  or not. Thus, when and if  $x_n$  turns out not to be a transcendental constant, we will still be able to escape the trap, by defining  $y_m$  to be a zero of some such  $q$ .

The preceding discussion provides an example of our main concern in the rest of the construction: guessing at the minimal differential polynomial  $p_{n_i, \rho}$  of  $x_{n_i}$  over the differential subfield  $\mathbb{Q}\langle \rho \rangle$ , where  $\rho = (x_{n_0}, \dots, x_{n_{i-1}})$  is the tuple of higher-priority elements of  $K$ , and then finding or adjoining a  $y_m \in F$  with  $h_s(n_i) = m$ , in such a way that if the guess  $p_{n_i, \rho}[s]$  at  $p_{n_i, \rho}$  turns out at some stage  $t > s$  to be incorrect, we can salvage the situation by redefining  $h_t(n_i)$  and  $h_t^{-1}(m)$ , without changing the structure  $F_t^0$  which we have built so far. The priority ranking serves as a mechanism to ensure that, for all  $m$  and  $n$ , eventually our approximations  $h_s(n)$  and  $h_s^{-1}(m)$  will indeed each converge to a limit.

The above discussion of transcendental constants also informs our use of Theorem 1.1 (as already seen at stage 0). While  $\delta X$  is known to be unconstrainable over  $\mathbb{Q}$ , there is no algorithm known for deciding constrainability of arbitrary differential polynomials  $p \in \mathbb{Q}\langle X \rangle$ ; the decidability of this set remains an open question. (In general, being constrainable is a  $\Sigma_2^0$  property, and it can be  $\Sigma_2^0$ -complete, as shown in [16].) So, while Lemma 4.3 can tell us the apparent minimal differential polynomial  $f$  of  $y_m$  over  $\mathbb{Q}\langle y_0, \dots, y_{m-1} \rangle$ ,



we will generally not know whether this  $f$  is constrainable or not. If it is, then  $K$ , being differentially closed, must contain a suitable preimage for  $y_m$ , which will eventually reveal itself. If  $f$  is not constrainable, however, then we run the risk of not finding a suitable preimage, which would leave  $\mathcal{S}_m$  unsatisfied. To guard against this, we use a Rabin embedding  $g_s$  of  $F_s$  into its computable differential closure  $\widehat{F}_s$ . This will map  $y_m$  to an element of  $\widehat{F}_s$  with  $f(g_s(y_0), \dots, g_s(y_m)) = 0$  there. If in fact  $f$  is not constrainable, then eventually  $g_s(y_m)$  will turn out to satisfy some polynomial of strictly lesser order over  $g_s(y_0), \dots, g_s(y_{m-1})$ , and when we see this, we will make  $y_m$  satisfy the same lower-order polynomial over  $y_0, \dots, y_{m-1}$  in  $F$ , so that this polynomial is the new minimal differential polynomial of  $y_m$ . Eventually, this will force  $y_m$  to satisfy a constrainable polynomial in  $F$ , allowing us to be sure that there is a suitable  $h$ -preimage for  $m$  among the indices  $n$  of elements of  $K$ . Thus we use the Rabin embedding and the differential closure  $\widehat{F}_s$  as a guide, to figure out what to do with  $y_m$  in case its current minimal differential polynomial is unconstrainable.

Now we give the algorithm to be followed at stage  $s+1$ , using the function  $h_s$  and the set  $U_s$  defined at stage  $s$ . The domain of  $h_s$  contains finitely many elements of  $\omega$ , which we view as indices of the elements  $x_n$  of  $K$ , while its range is viewed as a set of indices of elements  $y_m$  of the finite set  $F_s^0 = \{y_0, \dots, y_r\}$ . We order the indices of elements of  $F_s^0$  according to priority:

$$h_s(0) \prec 0 \prec h_s(1) \prec 1 \prec \dots \prec r,$$

and, after removing all repetitions from this list, we name these indices  $m_{0,s} \prec m_{1,s} \prec \dots$ . That is,  $m_{0,s} = h_s(0)$  is the highest-priority index, since keeping it fixed at subsequent stages will satisfy  $\mathcal{R}_0$ . The next highest-priority index is usually 0 itself, since  $\mathcal{S}_0$  wants  $h_s^{-1}(0)$  to stay fixed at all subsequent stages; however, if  $h_s(0) = 0$ , then  $m_{0,s} = 0$  and we do not repeat 0 in our list, but instead set  $m_{1,s} = h_s(1)$ , since  $\mathcal{R}_1$  has the next highest priority. If  $h_s(n)$  is undefined for some  $n$ , we simply skip that spot in our list of indices  $m_{i,s}$ . The list ends once it contains all indices of elements of  $F_s^0$  (namely  $\{0, 1, \dots, r\}$ ), by which point it must contain all indices in the range of  $h_s$ . For each  $i$ , we define  $n_{i,s} = h_s^{-1}(m_{i,s})$ , if this inverse image exists. For the least  $j$  such that  $n_{j,s}$  is not defined by this process, we set  $n_{j,s}$  to be the least element not in  $\text{dom}(h_s)$ , since we might be able to extend  $\text{dom}(h_{s+1})$  to include this element. Then, for each  $i \leq j$ , we set  $\rho_{i,s}$  to be the finite tuple  $(n_{0,s}, n_{1,s}, \dots, n_{i-1,s})$ .

Thus  $n_{0,s} = 0$ , and

$$n_{1,s} = \begin{cases} h_s^{-1}(0), & \text{if } h_s^{-1}(0) \neq 0 \\ 1, & \text{if } h_s^{-1}(0) = 0, \end{cases}$$

and so on. We have thereby ordered all elements of the domain of  $h_s$  according to the priority of the requirements they currently attempt to satisfy, with  $\rho_{i,s}$  containing those elements of higher priority than  $n_{i,s}$ .

We define  $\psi_s$  to be the (relational) atomic diagram of  $F_s^0 = \{y_0, \dots, y_r\}$  determined so far:

$$\psi_s(Y_0, \dots, Y_r) : \bigwedge_{i < j \leq r} Y_i \neq Y_j \ \& \ \bigwedge_{f \in U_s} f(Y_0, \dots, Y_k) = 0.$$

Similarly, for each  $i$  with  $n_{i,s}$  defined,  $\sigma_{i,s}$  is the current approximation to  $K$  up to  $x_{n_{i,s}}$ , using the priority ordering:

$$\sigma_i(X_{n_0}, \dots, X_{n_i})[s] : \bigwedge_{j \leq i} \left[ p_{n_j, \rho_j}(X_{n_0}, \dots, X_{n_j}) = 0 \ \& \ \bigwedge_{k < j} X_{n_k} \neq X_{n_j} \right] [s]$$

where, as defined earlier,  $p_{n_i, \rho_i}(X_{n_0}, \dots, X_{n_i})[s]$  is the current approximation to the minimal differential polynomial of  $x_{n_{i,s}}$  over  $\mathbb{Q}\langle \rho_{i,s} \rangle$ . (Having  $p_{n, \rho, s}$  be the zero polynomial when  $x_n$  appears to be differentially transcendental over  $\mathbb{Q}\langle \rho \rangle$  suits this definition of  $\sigma_{i,s}$  nicely.)

At stage  $s+1$ , we go through each  $\mathcal{R}_n$  and  $\mathcal{S}_n$  with  $n \leq s$  in turn, with one substage for each, starting with  $\mathcal{R}_0$ , then  $\mathcal{S}_0$ , then  $\mathcal{R}_1$ , etc. At the substage for a requirement  $\mathcal{R}_n$ , fix  $i$  such that  $n = n_{i,s}$ . (Such an  $i$  must exist, since we included the least index  $\notin \text{dom}(h_s)$  on our list of indices  $n_{i,s}$ . After this least element has been reached, no further substages will be executed at this stage.) Now we know that, for all  $n_{k,s}$  with  $k < i$ ,  $h_{s+1}(n_{k,s}) = h_s(n_{k,s})$ , since otherwise the stage would have ended before this substage. First we consult the theory **DCF**<sub>0</sub>, asking whether the sentence

$$\exists X_{n_0} \dots \exists X_{n_i} \sigma_i(X_{n_0}, \dots, X_{n_i})[s]$$

belongs to this theory. If not, then we do nothing at this substage, and do not go on to the next substage, but instead go straight to the final step of stage  $s+1$  (described below). In particular,  $h_{s+1}(n_{k,s})$  is undefined for all  $k \geq i$ . As a simple example, if  $p_{n_i, \rho_i} = X_{n_i} - a[s]$  and  $p_{n_j, \rho_j} = X_{n_j} - a[s]$  for the same rational  $a$  and for some  $j < i$ , then the sentence would be rejected as inconsistent. If it is consistent, then we follow these instructions.

1. If  $h_{s+1}(n_{i,s})$  has been defined at an earlier substage of stage  $s+1$ , then we keep that value and go on to the next substage. (This happens if  $h_{s+1}(n_{i,s}) = m$  for some  $m < n_{i,s}$ .)
2. If  $h_s(n_{i,s}) \downarrow$  and  $p_{n,\rho_i}(y_{h(n_0)}, \dots, y_{h(n_{i-1})}, X)[s]$  is the minimal differential polynomial of  $y_{h_s(n)}$  in  $F_s^0$  over  $\mathbb{Q}\langle y_{h(n_0)}, \dots, y_{h(n_{i-1})} \rangle[s]$  – for instance, if  $p_{n,\rho_i,s,s+1} = p_{n,\rho_i,s,s}$  – then we preserve the map, setting  $h_{s+1}(n) = h_s(n)$ , and go on to the next substage. (This includes the case where  $p_{n,\rho_i}[s]$  is the zero polynomial, i.e., where  $x_n$  currently appears to be differentially transcendental.) Lemma 4.3 allows us to find the minimal differential polynomial for  $y_{h_s(n)}$  in  $F_s^0$  and thus check whether this case holds.
3. Otherwise, either  $h_s(n)$  is undefined, or else  $h_s(n) = m'$  is defined with  $m' \geq n$  but  $p_{n,\rho_i}(y_{h(n_0)}, \dots, y_{h(n_{i-1})}, X)[s]$  is not the minimal differential polynomial of  $y_{m'}$  over  $\mathbb{Q}\langle y_{h(n_0)}, \dots, y_{h(n_{i-1})} \rangle[s]$  in  $F_s^0$ . (This latter case happens if  $p_{n,\rho_i}[s-1] \neq p_{n,\rho_i}[s]$ .) In this case,  $x_n$  *abandons* this  $y_{m'}$ , if it existed at all, and we will need to choose a new value  $m$  for  $h_{s+1}(n)$ . The element  $y_{m'}$  becomes *unattached*.

If  $p_{n,\rho}[s]$  is the zero polynomial, then we find the least number  $m > n$  such that  $y_m \notin F_s^0$  and define  $h_{s+1}(n) = m$ . This is the case where  $x_n$  currently appears to be differentially transcendental. This  $y_m$  is adjoined to  $F_{s+1}^0$ , with no change to  $U_{s+1}$  (so that  $y_m$  likewise appears to be differentially transcendental in  $F_s$ ).

If  $p_{n,\rho}[s+1]$  was nonzero, we search for the least  $m \leq r+1$  such that  $h_{s+1}^{-1}(m)$  is not yet defined and  $\mathbf{DCF}_0$  contains the sentence

$$\exists Y_0 \cdots \exists Y_{r+1} (\sigma_i(Y_{h(n_0)}, \dots, Y_{h(n_{i-1})}, Y_m) \ \& \ \psi(Y_0, \dots, Y_r))[s].$$

By induction on substages,  $\sigma_{i-1}(Y_{h(n_0)}, \dots, Y_{h(n_{i-1})})[s]$  is consistent with  $\psi_s(Y_0, \dots, Y_r)$ , and so, for some  $m \leq r+1$ ,  $\mathbf{DCF}_0$  must contain the above sentence. (If, for all  $m \leq r$ ,  $\mathbf{DCF}_0$  does not contain this sentence, then for  $m = r+1$  it must, because we checked already that  $\sigma_{i,s}$  lies in  $\mathbf{DCF}_0$  and because  $\psi_s$  contains no conditions at all on  $y_{r+1}$ .) Fixing this least  $m$ , we adjoin

$$p_{n,\rho_i}(Y_{h(n_0)}, \dots, Y_{h(n_{i-1})}, Y_m)[s]$$

to  $U_{s+1}$ ; this means that we are setting

$$p_{n,\rho_i}(y_{h(n_0)}, \dots, y_{h(n_{i-1})}, y_m) = 0[s]$$

in  $F$ , just as  $p_{n,\rho_i}(x_{n_0}, \dots, x_{n_{i-1}}, x_n) = 0[s]$  in  $K$ . With  $h_{s+1}(n) = m$ , this means that  $h_{s+1}$  still defines a (partial) isomorphism, based on the current approximation to  $K$ . If  $m = r + 1$ , we also add  $x_{r+1}$  to  $F_{s+1}^0$ .

No matter which case held here in item (3), we do not go on to the next substage, but instead continue to the final step of stage  $s + 1$  (described below).

This covers all the possibilities at substages dedicated to  $\mathcal{R}$ -requirements. Notice that, even if  $m$  lay in  $\text{range}(h_s)$  but not in  $\text{range}(h_{s+1})$ ,  $y_m$  is still in  $F_{s+1}^0$ , and  $U_s \subseteq U_{s+1}$ . This is necessary in order for  $F$  to be computable: once we have defined  $y_m$  to satisfy a polynomial in  $F$ , or not to, we must preserve that condition forever after. Eventually,  $\mathcal{S}_m$  will choose an  $h$ -preimage for  $m$  which will respect these conditions.

Next we explain the instructions for a substage for the requirement  $\mathcal{S}_m$ . We fix the  $i$  (which must exist) such that  $m_{i,s} = m$ , and the current minimal differential polynomial  $f$  of  $y_m$  over  $y_{m_0}, \dots, y_{m_{i-1}}[s]$ . Now either  $h_{s+1}^{-1}(m)$  has already been determined by some higher-priority  $\mathcal{R}_n$  (so  $\mathcal{S}_m$  has nothing to do), or  $h_s(n) = m$  for some  $n > m$ , or  $y_m$  is currently unattached (i.e.,  $m \notin \text{range}(h_s)$ ). In these latter two cases, it is not clear that we will ever be able to find any  $x \in K$  with minimal differential polynomial  $f$  over  $x_{n_0}, \dots, x_{n_{i-1}}[s]$ , since  $f$  might not be constrainable over these elements. (If  $h_s^{-1}(m) = n$  is defined, then  $x_n$  currently appears to fill this role, but in the noncomputable differential field  $K$ , this could change at any time.) So the requirement  $\mathcal{S}_m$  will search for some  $q \in \mathbb{Q}\{Y_0, \dots, Y_i\}$ , of strictly lower order in  $Y_i$  than the current minimal differential polynomial  $f$  of  $y_m$  over  $y_{m_0}, \dots, y_{m_{i-1}}[s]$ , such that it is consistent (with  $\mathbf{DCF}_0$  and also with the current atomic diagram of  $F_s^0$ ) for  $y_m$  to become a zero of  $q(y_{m_0}, \dots, y_{m_{i-1}}, Y_m)[s]$  as well. Of course, we cannot search through all possible  $q$  right at this substage, so the strategy is to use the zero  $g_s(y_m)$  of  $f$  in  $\widehat{F}_s$  as a guide. When extending  $F$  on behalf of lower-priority requirements, or in the final step, we always follow the dictates of  $\widehat{F}_s$  via the map  $g_s$ . If  $f$  really is the minimal differential polynomial of  $g_s(y_m)$  in the differential closure  $\widehat{F}_{i,s}$  of  $\mathbb{Q}\langle g(y_{m_0}), \dots, g(y_{m_{i-1}}) \rangle[s]$ , then we never find such a  $q$ , but in this case  $f$  is constrainable over that subfield, hence also constrainable over  $\mathbb{Q}\langle x_{n_0}, \dots, x_{n_{i-1}} \rangle[s]$ , so we must eventually find a value for  $h^{-1}(y_m)$  in the differentially closed field  $K$ . (If  $f$  has order 0, then the theory  $\mathbf{DCF}_0$  will ensure that we do not put more than  $\deg(f)$ -many roots of  $f$  into  $F$ , and all of them will eventually find preimages in  $K$ . If  $f$  has positive order, then it will have infinitely many zeroes in  $K$ , so we are

sure eventually to find one which can become  $h^{-1}(y_m)$ .) On the other hand, if we ever find that  $g_s(y_m)$  is a zero of a lower-order  $q$  over  $g(y_{m_0}), \dots, g(y_{n_{i-1}})$ , then it will be consistent for us to make  $y_m$  a zero of this  $q$  in  $F$  (as well as a zero of  $f$ , since we have already committed to having  $0 = f(y_{m_0}, \dots, y_{m_i})[s]$  in  $F$ ), and we will do so. If  $h_s^{-1}(m) = n$ , then  $\mathcal{R}_n$  has lower priority than  $\mathcal{S}_m$ , and so  $\mathcal{R}_n$  simply absorbs this injury and goes off to find another value for  $h(n)$ . (Eventually, if needed, it will set  $h_t(n) = m'$  for some  $m' \geq n$  at some later stage  $t$ , and then  $\mathcal{R}_n$  will have priority over  $\mathcal{S}_{m'}$  and will never again be injured in this way.) Of course, the lower-order  $q$  (if we find one) may not be constrainable either, but this process will continue, and ultimately we will reach the minimal differential polynomial of  $g_s(y_m)$  and will make  $y_m$  a zero of that polynomial in  $F$ , thus guaranteeing that our procedure will eventually find an  $h$ -preimage for  $y_m$  in  $K$ .

At a substage for a requirement  $\mathcal{S}_m$  within stage  $s + 1$ , we follow these instructions. Fix the unique  $i$  such that  $m = m_{i,s}$ .

1. If there exists an  $n \leq m$  such that  $h_{s+1}(n)$  has already been defined to equal  $m$ , then we go on to the next substage.
2. If  $h_s^{-1}(m)$  was defined and equal to some  $n = n_{i,s} > m$ , and  $p_{n,\rho_i}[s] \neq p_{n,\rho_i}[s-1]$ , then  $y_m$  has been *abandoned* by this  $x_n$  and has become *unattached*. We leave  $h_{s+1}^{-1}(m)$  undefined, and, instead of continuing to the next substage, we continue with the final step of stage  $s + 1$  (described below).
3. Otherwise, either  $h_s^{-1}(m)$  was undefined (and  $h_{s+1}^{-1}(m)$  has not been defined at an earlier substage of this stage), or  $h_s^{-1}(m) = n$  for some  $n = n_{i,s} > m$  with  $p_{n,\rho_i}[s] = p_{n,\rho_i}[s-1]$ . Now  $y_m$  has the option to change its minimal differential polynomial over  $\mathbb{Q}\langle y_{m_0}, \dots, y_{m_{i-1}} \rangle[s]$  and abandon this  $x_n$  (if it exists). We search through the first  $s$  differential polynomials  $q \in \mathbb{Q}\{Y_0, \dots, Y_i\}$ , whose rank in  $Y_i$  is strictly less than the rank of the apparent minimal differential polynomial of  $y_m$  in  $F_s^0$  over  $\mathbb{Q}\langle y_{m_0}, \dots, y_{m_{i-1}} \rangle[s]$ . (Lemma 4.3 allows us to determine the apparent minimal polynomial. If it is the zero polynomial, then  $q$  is allowed to have any rank at all.) Apply Ritt's reduction algorithm to those among them (if any) which have  $0 = q(g(y_{m_0}), \dots, g(y_{m_i}))[s]$  in  $\widehat{F}_s$ . (Here we use the function  $g_s$  defined at the end of stage  $s$ .) Thus we may assume that our  $q$  has the least possible rank in  $Y_i$ . Moreover, if it were reducible, then one of its irreducible factors would also have

$g(y_{m_0}), \dots, g(y_{m_i})[s]$  as a zero, and we can identify that factor using the splitting algorithm for this field. Thus we may assume that the  $q$  we found is irreducible and monic, with  $g(y_{m_0}), \dots, g(y_{m_i})[s]$  as a zero.

Now, if we found such a  $q$ , then  $\widehat{F}_s$  shows that it is consistent with  $\mathbf{DCF}_0$  for  $y_m$  to become a zero of this  $q$ , and we add  $q(Y_{m_0}, \dots, Y_{m_i})[s]$  to  $U_{s+1}$ , leaving  $h_{s+1}(n)$  and  $h_{s+1}^{-1}(m)$  both undefined. This changes  $F$ : the apparent minimal differential polynomial of  $y_m$  will now be  $q(y_{m_0}, \dots, y_{m_{i-1}}, Y_m)[s]$ . Of course,  $y_m$  is still a root of the previous apparent minimal polynomial (otherwise  $F$  would not be a computable differential field); only the minimality has changed. The element  $y_m$  becomes *unattached* (as does  $x_n$ , if it existed; it has been *abandoned* by  $y_m$ ). Instead of going on to the next substage, we now follow the instructions given below for unattached  $y_m$ .

If  $0 \neq q(g(y_{m_0}), \dots, g(y_{m_i}))[s]$  for all the differential polynomials  $q$  we examined, then we either keep  $h_{s+1}^{-1}(m) = h_s^{-1}(m)$  and go on to the next substage (if  $h_s^{-1}(m)$  was defined), or else skip all remaining substages and execute the instructions for an unattached  $y_m$  (if  $h_s^{-1}(m)$  was undefined).

The instructions to be executed for an unattached  $y_m$  are straightforward: essentially, the task is to check whether some  $x_n$  can be found for which it is consistent to define  $h_{s+1}(n) = m$ . (Notice that, at this stage  $s + 1$ , we execute these instructions for only one unattached  $y_m$ , by finding the least  $i$  for which  $y_{m_{i,s}} \notin \text{range}(h_{s+1})$  and setting  $m = m_{i,s}$ .) Consider in turn each  $n \leq s$  with  $n \notin \{n_{0,s}, \dots, n_{i-1,s}\}$ , and check whether the sentence

$$\exists Y_0 \dots \exists Y_r \left[ \begin{array}{l} \psi(Y_0, \dots, Y_r) \ \& \ \sigma_i(Y_{h(n_0)}, Y_{h(n_1)}, \dots, Y_{h(n_{i-1})}) \\ \& \ p_{n, \rho_i}(Y_{h(n_0)}, Y_{h(n_1)}, \dots, Y_{h(n_{i-1})}, Y_m) = 0 \end{array} \right] [s]$$

lies in  $\mathbf{DCF}_0$ . (By our choice of  $i$ ,  $n_{i-1,s} = h_s^{-1}(m_{i-1,s})$  must be defined, and so  $\rho_{i,s} = (n_{0,s}, \dots, n_{i-1,s})$  exactly as defined for the  $\mathcal{R}$ -substages.) If it lies in  $\mathbf{DCF}_0$ , then we define  $h_{s+1}(n) = m$ ; if not, then we go on to the next  $n$ . If this fails for all of these (finitely many)  $n$ , then  $h_{s+1}^{-1}(m)$  remains undefined. In either case, we do not go on to any other substage, but instead execute the final step for stage  $s + 1$ .

**Final Step.** To finish stage  $s + 1$ , after the last substage has been completed, we must define  $\widehat{F}_{s+1}$  and the embedding  $g_{s+1}$  of  $F_{s+1}^0$  into  $\widehat{F}_{s+1}$ , and must also take one more step towards closing  $F$  under the differential field

operations. First we consider  $\widehat{F}_{s+1}$ . Find the greatest  $i$  such that  $h_{s+1}^{-1}(m_{j,s}) = h_s^{-1}(m_{j,s})$  for all  $j \leq i$ . For each  $j \leq i$ , let  $g_{s+1}(y_{m_{j,s}}) = g_s(y_{m_{j,s}})$ , and let  $\widehat{F}_{j,s+1} = \widehat{F}_{j,s} \subseteq \widehat{F}_s$  be the differential closure of  $\mathbb{Q}\langle g(y_{m_0}), \dots, g(y_{m_j}) \rangle[s]$ , as previously defined. By induction, these were all built with  $\widehat{F}_{j-1,s} \subseteq \widehat{F}_{j,s}$ . Then use the following procedure to extend this  $g_{s+1}$  to  $y_{m_{i+1},s}$ , then to  $y_{m_{i+2},s}$ , and so on until we reach  $y_r$ .

If we have extended  $g_{s+1}$  to  $y_{m_{j,s}}$ , write  $m = m_{j+1,s}$ . Use Lemma 4.3 to find the apparent minimal differential polynomial  $f(y_{m_0}, \dots, y_{m_i}, Y)[s]$  of  $y_m$  over  $\mathbb{Q}\langle y_{m_0}, \dots, y_{m_i} \rangle[s]$ .

- If this  $f$  is the zero polynomial, and  $h_{s+1}^{-1}(m)$  is defined and equal to some  $n = n_{j,s} \leq m$ , then adjoin to  $\widehat{F}_{j,s+1}$  a differential transcendental  $t$ , use Theorem 1.1 to build the differential closure  $\widehat{F}_{j+1,s+1}$  of  $\widehat{F}_{j,s+1}\langle t \rangle$  (viewing the latter as a subfield of the former, via the Rabin embedding given by Theorem 1.1), and set  $g_{s+1}(y_m) = t$  within this  $\widehat{F}_{j+1,s+1}$ .
- If  $f$  is not the zero polynomial and has order  $> 0$ , and  $h_{s+1}^{-1}(m)$  is defined and equal to some  $n = n_{j,s} \leq m$ , then by the construction

$$f(Y_{m_0}, \dots, Y_{m_{j+1}}) = p_{n,\rho_j}(Y_{h(n_0)}, \dots, Y_{h(n_{j+1})})[s].$$

Now we adjoin to  $\widehat{F}_{j,s+1}$  a new element  $z$ , which we define to be a zero of  $f(g(y_{m_0}), \dots, g(y_{m_j}), Y_{m_{j+1}})[s]$ , and we let  $\widehat{F}_{j+1,s+1}$  be the differential closure of  $\widehat{F}_{j,s+1}\langle z \rangle$ , using Theorem 1.1 and regarding  $\widehat{F}_{j,s+1}$  as a differential subfield of  $\widehat{F}_{j,s+1}$  via the Rabin embedding. Fix  $g_{s+1}(y_m) = z$  in  $\widehat{F}_{j+1,s+1}$ .

(These first two items cover the situation in which the element  $x_n$  of  $K$  has higher priority than its  $h_{s+1}$ -image  $y_m$ . Since in general we do not know whether the current approximation  $p_{n,\rho_j}[s]$  to its minimal differential polynomial is constrainable or not, we need to put a special element  $z$  into  $\widehat{F}$ , as  $\widehat{F}_{j,s+1}$  may not have contained any element with this minimal differential polynomial. If in fact  $p_{n,\rho_j}[s]$  was constrainable, then it already had infinitely many zeroes in  $\widehat{F}_{j,s+1}$  (since its order was positive), and adjoining one more does not change matters: in this case  $\widehat{F}_{j,s+1}$  was already a differential closure of  $\mathbb{Q}\langle y_{m_0,s+1}, \dots, y_{m_{j,s+1}} \rangle$ , and  $\widehat{F}_{j+1,s+1}$  will just be another differential closure thereof.)

- If  $f$  has order 0, (i.e., it is just an algebraic polynomial, of some degree  $d$ ), then find all  $d$  roots of  $f(g(y_{m_0}), \dots, g(y_{m_j}), Y_m)[s]$  in  $\widehat{F}_{j,s+1}$ . Since  $\psi_s(y_0, \dots, y_r)$  is consistent with  $\mathbf{DCF}_0$ , at least one of these roots in the differentially closed field  $\widehat{F}_{j,s+1}$  can serve as  $g_{s+1}(y_m)$ ; we can identify such a root  $z$  because it must also make every polynomial in  $U_{s+1} \cap \mathbb{Q}\{Y_{m_0}, \dots, Y_{m_{j+1}}\}[s]$  equal 0 in  $\widehat{F}_{j,s+1}$  when the tuple  $(g(y_{m_0}), \dots, g(y_{m_j}), z)[s]$  is plugged into it. We set  $g_{s+1}(y_m)$  to be the first  $z$  we find which does so.
- If  $f$  has order  $> 0$ , and  $h_{s+1}^{-1}(m)$  is either undefined or equal to some  $n > m$ , then set  $\widehat{F}_{j+1,s+1} = \widehat{F}_{j,s+1}$ . (This includes the case where  $f$  is the zero polynomial and  $x_m$  is unattached.) Now we simply search for some  $z \in \widehat{F}_{j,s+1}$  which makes every polynomial in  $U_{s+1} \cap \mathbb{Q}\{Y_{m_0,s}, \dots, Y_{m_{j+1},s}\}$  equal 0 when the tuple  $(g(y_{m_0}), \dots, g(y_{m_j}), z)[s]$  is plugged into it.  $\widehat{F}_{j,s+1}$  must contain some  $z$  satisfying this condition (and in particular  $0 = f(g(y_{m_0}), \dots, g(y_{m_j}), z)[s]$ ), by existential closure of the differential closure  $\widehat{F}_{j,s+1}$ , and when we find one, we declare it to be  $g_{s+1}(y_m)$ .

(This is the situation in which  $y_m$  either is currently unattached, or else has priority over  $x_{h_{s+1}^{-1}(m)}$ . Notice that  $h_{s+1}^{-1}(m) \neq h_s^{-1}(m)$ , since  $j+1 > i$ , meaning that this value of  $h_{s+1}^{-1}(m)$  was just defined at this very stage. We cannot be certain whether its apparent minimal differential polynomial  $f$  is constrainable, but by mapping  $y_m$  to an element of  $\widehat{F}_{j,s+1}$ , we ensure that  $y_m$  will ultimately be constrained over  $\mathbb{Q}\langle y_{m_0}, \dots, y_{m_j} \rangle[s]$ : if  $f$  is not constrainable, then no element of  $\widehat{F}_{j,s+1}$  can have minimal differential polynomial  $f$ , so the  $z = g_{s+1}(y_m)$  we chose will turn out to be a zero of some lower-rank polynomial than  $f$ , and item (3) of the  $\mathcal{S}_m$  substage will eventually cause  $y_m$  to become a zero of that polynomial as well. Indeed, since  $z$  must have a constrainable minimal differential polynomial,  $y_m$  will as well, as this process continues. Therefore, there will be preimages available for  $y_m$  in  $K$ : either it will turn out to be algebraic over  $\mathbb{Q}$ , or it will have a constrainable minimal differential polynomial of positive order, which must have infinitely many realizations in the differentially closed field  $K$ .)

Having completed this process, we set  $\widehat{F}_{s+1} = \cup_{j \leq r} \widehat{F}_{j,s+1}$ . Now we take a step to close  $F$  under the differential field operations.



At an even stage  $s + 1 = 2t$ , consider the  $t$ -th differential polynomial  $q(Y_0, \dots, Y_k)$  in a fixed computable enumeration of  $\mathbb{Q}\{Y_0, Y_1, \dots\}$ . Let  $F'_{s+1} = \{y_0, \dots, y_{r'}\}$  be the elements enumerated into  $F$  up till this point (so either  $r' = r$  or  $r' = r + 1$ , as at most one element has been added to  $F_{s+1}^0$  so far in this stage), and let  $\psi'_{s+1}$  be the formula defined exactly as  $\psi_s$  was, only using  $F'_{s+1}$  and  $r'$  and the set  $U_{s+1}$  defined so far in this stage. Find the element  $z = q(g_{s+1}(y_0), \dots, g_{s+1}(y_{r'}))$  in  $\widehat{F}_{s+1}$ . If this  $z$  lies in the image of  $F'_{s+1}$  under  $g_{s+1}$ , then  $z = g_{s+1}(y_m)$  for some  $m \leq r'$ , and we enumerate  $(q - Y_m)$  into  $U_{s+1}$ . If not, then we set  $m = r' + 1$ , enumerate  $y_m$  into  $F_{s+1}^0$  and  $(q - Y_m)$  into  $U_{s+1}$ , and set  $g_{s+1}(y_m) = z$ . In either case, the differentially closed field  $\widehat{F}_{s+1}$  shows that our action is consistent with  $\mathbf{DCF}_0$ . Thus we have ensured that there exists an  $m$  with  $(q - Y_m) \in U$ .

At an odd stage  $s + 1 = 2t - 1$ , we move to ensure that  $F$  be closed under inversion. Define  $F'_{s+1}$ ,  $r'$ , and  $\psi'_{s+1}$  just as above. For the element  $y_t$ , find the element  $z = (g_{s+1}(y_t))^{-1}$  in  $\widehat{F}_{s+1}$ . If  $z = g_{s+1}(y_m)$  for some  $m \leq r'$ , then enumerate  $(Y_t Y_m - 1)$  into  $U_{s+1}$ . If not, then we set  $m = r' + 1$ , enumerate  $y_m$  into  $F_{s+1}^0$  and  $(Y_t Y_m - 1)$  into  $U_{s+1}$ , and set  $g_{s+1}(y_m) = z$ . Again, the differentially closed field  $\widehat{F}_{s+1}$  shows that our action is consistent with  $\mathbf{DCF}_0$ . (Notice that we did not execute this step for the zero element  $y_0$  of  $F$ , which needs no multiplicative inverse.)

This completes stage  $s + 1$ , and ends the construction. We define  $F = \{y_m : m \in \omega\}$ , but the important objects constructed were the set  $U = \cup_s U_s$ , which will define the differential field operations effectively on  $F$ , and the finite functions  $h_s$ , whose limit will be the isomorphism from  $K$  onto  $F$ . We remark here that, every time any differential polynomial  $f(Y_0, \dots, Y_{k_f})$  (for any  $m$ ) was enumerated into  $U_s$ , we checked first to ensure that the finite conjunction

$$\exists Y_0 \cdots \exists Y_r \bigwedge_{f \in U_s} 0 = f(Y_0, \dots, Y_{k_f})$$

belonged to the theory  $\mathbf{DCF}_0$ . (In some cases, this check was accomplished by confirming that the conjunction held of the elements  $g_s(y_0), \dots, g_s(y_r)$  in the computable differentially closed field  $\widehat{F}_s$  in use at that stage.) It follows that the entire set of these conjunctions, for all  $s$ , is consistent with  $\mathbf{DCF}_0$ .

Now we show that in the structure  $F$  constructed above, the operations of addition, multiplication, and differentiation are in fact computable. The construction built the domain  $\{y_0, y_1, \dots\}$  of  $F$  and, for every differential polynomial  $f \in \mathbb{Q}\{Y_0, \dots, Y_n\}$  (for every  $n$ ), determined at some finite stage

$s$  some  $m$  for which  $(f - y_m) \in U$ . For this  $f$ , this  $m$  is unique, because we checked at every step for consistency with  $\mathbf{DCF}_0$ , including the condition that  $y_m \neq y_{m'}$  for all  $m \neq m'$ . Since  $U$  is c.e., we can therefore decide membership in  $U$ :  $f \notin U$  iff there exists  $m > 0$  with  $(f - y_m) \in U$ . Now, to compute  $y_i + y_j$ , just find the polynomial  $Y_i + Y_j$  in our list of differential polynomials, search for the unique  $m$  such that  $(y_i + y_j - y_m) \in U$ , and conclude that  $y_i + y_j = y_m$  in  $F$ . The differential polynomials  $Y_i Y_j$  and  $\delta Y_i$  likewise allow computation of multiplication and differentiation in  $F$ , and so  $F$  is indeed a computable structure. Moreover, for every  $t$ , there is some  $m$  such that  $(Y_t + Y_m) \in U$ , since the polynomial  $(-Y_t)$  was assigned a value  $Y_m$  in the final step at the even stage when we considered this polynomial. Hence  $F$  is closed under negation, and at the odd stages we ensured that it is also closed under inversion.

Now that we know we can compute the operations (and that  $F$  is closed under these operations and under negation and inversion), the fact that this computable structure  $F$  really is a differential field follows from the consistency of  $U$  with  $\mathbf{DCF}_0$ , since the axioms for a differential field (with identity elements  $y_0$  and  $y_1$ ) are all universal axioms, except for the existence of inverses. The fact that  $F$  is isomorphic to  $K$  as a differential field (and hence is differentially closed) will follow from these claims:

- for every  $i$ ,  $n_i = \lim_s n_{i,s}$  exists, and the map  $i \mapsto n_i$  is a permutation of  $\omega$ ;
- for every  $i$ ,  $m_i = \lim_s m_{i,s}$  exists, and the map  $i \mapsto m_i$  is a permutation of  $\omega$ ;
- the function  $h = \lim_s h_s$  is a bijection from  $\omega$  onto  $\omega$ , and hence defines a bijection  $x_n \mapsto y_{h(n)}$  from  $K$  onto  $F$ ; and
- for every  $i$ , the polynomial  $p_i = \lim_s p_{n_{i,s}, \rho_{i,s}, s} \in \mathbb{Q}\{X_{n_0}, X_{n_1}, \dots, X_{n_i}\}$ , exists, and  $p_i(Y_{h(n_0)}, \dots, Y_{h(n_i)}) \in U$  and no  $q(Y_{h(n_0)}, \dots, Y_{h(n_i)})$  in  $U$  has lower  $Y_{h(n_i)}$ -rank than  $p_i$ . (Here  $\rho_i = (n_0, \dots, n_{i-1}) = \lim_s \rho_{i,s}$ , from the first claim.)

The first three of these claims can be proven as a group by a single induction. (Now that we are considering limits such as  $m_i = \lim_s m_{i,s}$ , we will abandon the  $[s]$  notation, to avoid confusion.)

**Lemma 4.4** *For every  $m$ , there exists a unique  $i$  with  $\lim_s m_{i,s} = m$ ; likewise, for every  $n$ , there exists a unique  $i$  with  $\lim_s n_{i,s} = n$ . Moreover, every requirement  $\mathcal{R}_n$  and  $\mathcal{S}_m$  is satisfied by the foregoing construction.*

*Proof.* The uniqueness of  $i$ , for any single  $m$  or  $n$ , is immediate from our definitions of  $m_{i,s}$  and  $n_{i,s}$ . We specifically excluded all repetitions from the first sequence, making  $m_{i,s} \neq m_{j,s}$  for every  $i < j$ . Moreover, every  $h_s$  is injective, and so  $n_{i,s} = h_s^{-1}(m_{i,s}) \neq h_s^{-1}(m_{j,s}) = n_{j,s}$  for all  $i < j$  as well. (Recall that by our definition, every  $n_{i,s}$  lies in  $\text{dom}(h_s)$ . The injectivity of each  $h_s$  follows from its construction: we always included in  $\psi_s(Y_0, \dots, Y_r)$  the conditions that  $Y_i \neq Y_j$  for all  $i < j \leq r$ , and similarly in  $\sigma_{i,s+1}$  that  $X_{n_{i,s}} \neq X_{n_{j,s}}$ , and then we required the choice of each new  $h_{s+1}(n)$  to have  $\sigma_{i,s+1}(Y_{h_{s+1}(n_{0,s})}, \dots, Y_{h_{s+1}(n_{i-1,s})}, Y_{h_{s+1}(n)})$  consistent with  $\psi_s(Y_0, \dots, Y_r)$ .)

We proceed by induction on these requirements, according to their priority order, starting with  $\mathcal{R}_0$ . The inductive hypothesis is that there exists a stage  $s_0$  and (unique) numbers  $j$  and  $k$  such that, for every  $s \geq s_0$  and each higher-priority requirement  $\mathcal{R}_{n'}$  or  $\mathcal{S}_{m'}$ ,  $n_{j,s} = n'$  and  $m_{k,s} = m'$  and  $h_s(n') = h_{s_0}(n')$  and  $h_s^{-1}(m') = h_{s_0}^{-1}(m')$ . Turning to the minimal polynomials in  $K$ , we may also assume that  $s_0$  is so large that, for every  $n' = n_{j,s} < n$ ,  $p_{n', \rho_{j,s}, s} = p_{n', \rho_{j,s}, s_0}$  (noting that  $\rho_{j,s} = \rho_{j,s_0}$  by the previous part of the hypothesis). That is, all approximations to minimal polynomials of higher-priority elements of  $K$  have converged by stage  $s_0$ . It follows that, from stage  $s_0 + 1$  on, every substage for a higher-priority requirement will do nothing except to go on to the next substage. Moreover, for every  $s \geq s_0$  and each higher-priority  $m'$ , this ensures that  $g_{s_0}(y_{m'}) = g_s(y_{m'})$ , according to the definitions of  $\widehat{F}_s$  and  $g_{s+1}$ , which preserve the values of  $g_s$  on those  $y_{m_{i,s}}$  with  $h_{s+1}^{-1}(m_{i,s}) = h_s^{-1}(m_{i,s})$ , and make the differentially closed subfield generated by these  $g_s(y_{m'})$  a subfield of  $\widehat{F}_{s+1}$ .

Suppose this inductive hypothesis holds of every requirement of higher priority than  $\mathcal{R}_n$ . If there exists an  $m < n$  with  $h_{s_0}(n) = m$ , then the satisfaction of  $\mathcal{S}_m$  shows that  $\mathcal{R}_n$  is satisfied as well. So assume that there is no such  $m$ . If  $h_{s_0}(n)$  is undefined, then at stage  $s_0 + 1$  the construction will reach the substage for  $\mathcal{R}_n$  and will act according to item (3) at that substage, and will choose a value  $h_{s_0+1}(n) \leq r + 1$ . This  $y_{h_{s_0+1}(n)}$  therefore lies in  $F_s^0$  at all  $s \geq s_0 + 1$ . At the next stage  $s_0 + 2$ ,  $n$  will lie in the domain of  $h_{s_0+1}$ , and therefore will have  $n = n_{i,s_0+1}$  for some  $i$ , i.e.,  $n$  will have been assigned a priority, corresponding to the requirement  $\mathcal{R}_n$ . It is possible that  $h_s(n)$  will become undefined at subsequent stages, as we discuss below, but each

time it does, it will again be redefined to equal  $n_{i,s}$  at the following stage, *for the exact same  $i$* , since hereafter no element except  $x_{0,s_0}, \dots, x_{i-1,s_0}$  ever has higher priority than  $n$ , and all of those elements are fixed at all subsequent stages. Therefore, once we have shown that  $\lim_s h_s(n)$  converges, we will have established that there does exist an  $i$  with  $n = n_i = \lim_s n_{i,s}$ .

Let  $\rho = \rho_{i,s_0+1}$  be the sequence of indices of elements in  $K$  of higher priority than  $n$ . This too never changes at stages  $> s_0$ . But now the approximations  $p_{n,\rho,s}$  to the minimal differential polynomial of  $x_n$  over  $\mathbb{Q}\langle x_0, \dots, x_{i-1} \rangle$  (with  $x_j = \lim_s x_{j,s}$ ) must converge, to some limit  $p_n(X_0, \dots, X_i)$ . Let  $s_1 > s_0$  be a stage by which this convergence has occurred. From then on, item (2) in the substage for  $\mathcal{R}_n$  will always apply, and so  $h_s(n)$  will never again change its value. Thus the requirement  $\mathcal{R}_n$  is indeed satisfied, and the existence of the (unique)  $i$  with  $n = n_i = \lim_s n_{i,s}$  follows.

Now we turn to the inductive step for a requirement  $\mathcal{S}_m$ , using the stage  $s_0$  defined above by the inductive hypothesis on all higher-priority requirements. Once again, it follows that every higher-priority requirement will do nothing at its substage during each stage  $> s_0$ , and so the  $\mathcal{S}_m$ -substage will be reached at every such stage. If  $h_{s_0}(n) = m$  for some  $n \leq m$ , then the satisfaction of the higher-priority requirement  $\mathcal{R}_n$  shows that  $m = \lim_s h_s(n)$ ; so assume that this is not the case. Notice that  $F_s^0$  increases at infinitely many stages  $s$  (because the final steps of the stages yield closure of  $F$  under addition, with characteristic 0, for instance), so eventually some  $F_{s_1}^0$  will include the element  $y_m$ . At this point, an  $i$  will be chosen for which  $m_{i,s} = m$ , since this happens for all indices of elements of  $F_s^0$ . Moreover, taking  $s_1 > s_0$ , we know that the higher-priority requirements never act again, we will in fact have  $m_{i,s} = m$  at all stages  $> s_1$  as well; this proves the existence of the  $i$  with  $m = m_i = \lim_s m_{i,s}$ , and its uniqueness was already shown.

If  $s > s_1$  and  $h_s^{-1}(m)$  is undefined, then the construction at stage  $s+1$  will reach item (3) in the  $\mathcal{S}_m$ -substage. Now as shown earlier, the inductive hypothesis ensures that  $g \upharpoonright \{y_{m_0}, \dots, y_{m_{i-1}}\}$  stays constant hereafter, with its image in the same fixed, differentially closed subfield of  $\widehat{F}_s$ . The final step of this stage will define  $g_s(y_m)$  to be some element  $z \in \widehat{F}_s$  within the differential closure of  $\{g_s(y_0), \dots, g_s(y_{m_{i-1}})\}$ , consistent with the facts about  $y_m$  already enumerated in  $F$  via  $U_{s+1}$ . As long as  $h_s^{-1}(m)$  remains undefined, this value of  $g_s(y_m)$  will be preserved in each subsequent  $\widehat{F}_s$ . Eventually, in item (3) of the  $\mathcal{S}_m$  substage at some stage  $s$ , the construction will notice that  $g_s(y_m)$  is a zero of some differential polynomial over  $\mathbb{Q}\langle g_s(y_{m_0}), \dots, g_s(y_{m_{i-1}}) \rangle$ ,

(since every element of the differential closure of  $\{g_s(y_0), \dots, g_s(y_{m_{i-1}})\}$  satisfies some such differential polynomial), and will enumerate this polynomial into  $U_{s+1}$ . Indeed, eventually the construction will do this with the minimal differential polynomial  $f$  of  $g_s(y_m)$  over  $\mathbb{Q}\langle g_s(y_{m_0}), \dots, g_s(y_{m_{i-1}}) \rangle$ , and from then on,  $f$  will be the minimal differential polynomial of  $y_m$  over  $\mathbb{Q}\langle y_{m_0}, \dots, y_{m_{i-1}} \rangle$  in  $F$  as well. Moreover, since this  $f$  is the minimal differential polynomial of  $g_s(y_m)$ , and since all elements of the differential closure of  $\mathbb{Q}\langle g_s(y_0), \dots, g_s(y_{m_{i-1}}) \rangle$  are constrained over this subfield,  $f$  must be constrainable. But  $K$  is also a differentially closed field, so  $K$  must contain elements satisfying the same constrained pair over  $x_{n_0}, \dots, x_{n_{i-1}}$  – indeed, infinitely many such elements, unless  $f$  has order 0. Therefore, assuming  $f$  is of positive order, the construction will follow the instructions for the unattached element  $y_m$  until, at some stage, it finds an element of  $K$  which (according to the computable approximation of  $K$ ) has  $f$  as its minimal differential polynomial over  $x_{n_0}, \dots, x_{n_{i-1}}$ . If this approximation subsequently changes, then the construction will search again, and since it always chooses  $n_i$  to be least such that  $x_{n_i}$  appears (at the current stage) to have minimal differential polynomial  $f$ , eventually it will settle on the least  $n$  for which  $f$  really is the minimal differential polynomial in  $K$  over  $y_{n_0}, \dots, y_{n_{i-1}}$ . This  $n$  will then be defined to be  $h_s^{-1}(m_i)$ , and will never change again (so this  $n$  will be  $n_i = \lim_s n_{i,s}$ ). Thus  $\mathcal{S}_m$  will be satisfied, in the case where  $f$  has positive order.

For an  $f$  of order 0, and of degree  $d$ , the situation is different. Now  $F$ , being consistent with  $\mathbf{DCF}_0$ , contains at most  $d$  roots of  $f$  at each stage, of which  $y_m$  is one. Therefore, only  $(d-1)$  of these roots have priority over  $y_m$ . But the differentially closed field  $K$  must also contain exactly  $d$  roots of this  $f$ , and with at most  $(d-1)$  of them controlled by higher-priority elements  $y_{m'}$ , at least one of them, say  $x_n$ , will remain available to serve as  $h_s^{-1}(m_i)$ . Eventually the computable approximations will reveal this one and settle on it, and this  $n$  will be defined to be  $h_s^{-1}(m_i)$ . Thereafter (assuming the approximation has converged), we will have  $h_s^{-1}(m_i) = n$  for all subsequent  $s$ , and so again  $\mathcal{S}_m$  is satisfied. This completes the proof of the lemma. ■

Finally we consider the last claim, for a fixed  $i$ . We have seen above that the limit  $n_j = \lim_s n_{j,s}$  exists for every  $j$ , and so we may set  $\rho_i = \lim_s \rho_{i,s}$ . The computable approximations to minimal differential polynomials in  $K$  all converge, so the limiting polynomial  $p_i = \lim_s p_{n_i, \rho_{i,s}, s}$  must exist, and we have also seen above that  $m_i = h(n_i) = \lim_s h_s(n_i)$  exists.

If  $m_i < n_i$ , then the reason why  $\mathcal{S}_{m_i}$  chose to make and keep  $h_s^{-1}(m_i) = n_i$  is that  $p_i(y_{m_0}, \dots, y_{m_i}) \in U$  (as seen in the instructions for unattached  $y_m$ , where  $n_i$  was first chosen) and that item (3) of the  $\mathcal{S}_{m_i}$ -substage never found a differential polynomial of lower rank for which  $g(y_{m_i})$  was a zero. Thus, this  $p_i$  was the minimal differential polynomial for  $g(y_{m_i})$  over  $g(y_{m_0}), \dots, g(y_{m_{i-1}})$ , just as it is for  $x_{n_i}$  over  $x_{n_0}, \dots, x_{n_{i-1}}$ . Because the construction used  $\widehat{F}_s$  as a guideline, it will not ever have made  $y_{m_i}$  into a zero of any differential polynomial of lower rank either. This is exactly what the claim requires.

On the other hand, if  $m_i \geq n_i$ , then it was  $\mathcal{R}_{n_i}$  which chose to make and keep  $h_s(n_i) = m_i$ . The instructions for the  $\mathcal{R}_{n_i}$  substages show that this was done because the apparent minimal differential polynomial of  $y_{m_i}$  over  $y_{m_0}, \dots, y_{m_{i-1}}$  was  $p_i$  itself. (It may be that  $\mathcal{R}_{n_i}$  added a new element  $x_{m_i}$  to  $F$  specifically to satisfy  $p_i$  and to be the image of  $y_{n_i}$ ; or it may have found an existing element of  $F$  which sufficed. The latter case might well hold if  $p_i$  has order 0, for example, and if the full complement of roots of  $p_i$  had been added to  $F$  before  $p_{n_i, \rho_i, s, s}$  converged to  $p_i$ .) Moreover, with  $n_i \geq m_i$ ,  $\mathcal{R}_{n_i}$  had priority over this  $y_{m_i}$  (once all higher-priority requirements had finished acting), and so the definition of  $g_s(y_{m_i})$  made it a zero of  $p_i$  over  $g_s(y_{m_0}), \dots, g_s(y_{m_{i-1}})$  in  $\widehat{F}_s$  and kept  $p_i$  as its minimal differential polynomial there; this is the content of the first two items in the recursive construction of  $g_{s+1}$ . Therefore, the construction never found any lower-rank differential polynomial for which to make  $y_{m_i}$  a zero, and so this  $y_{m_i}$  does indeed have the same minimal differential polynomial  $p_i$  over  $y_{m_0}, \dots, y_{m_{i-1}}$  that  $x_{n_i}$  has over  $x_{n_0}, \dots, x_{n_{i-1}}$ . This completes the proof of the final claim, and the theorem follows.  $\blacksquare$

Theorem 4.1 will remind many readers of the well-known theorem of Downey and Jockusch from [4], that every low Boolean algebra has a computable copy. In fact, though, the parallels between these results are few. The theorem for Boolean algebras has been extended to included  $\text{low}_4$  Boolean algebras, in work by Thurber [29] and Knight and Stob [13], whereas Corollary 3.3 shows that the result for  $\mathbf{DCF}_0$  does not even extend to the  $\text{low}_2$  case. Moreover, the proof of Theorem 4.1 constructed a  $\Delta_2^0$ -isomorphism from the low model of  $\mathbf{DCF}_0$  to its computable copy, whereas for Boolean algebras, there is always a  $\Delta_3^0$ -isomorphism but not always a  $\Delta_2^0$  one (as noted earlier). The construction here for relied heavily on the completeness and decidability of the theory  $\mathbf{DCF}_0$ , whereas the theory of Boolean algebras is certainly not complete. Also, for Boolean algebras we do not have any analogue of Harring-

ton's theorem (Theorem 1.1), which was heavily used here. Conversely, the construction in [4] uses theorems of Vaught and Rummel which are specific to Boolean algebras and have no obvious analogue for differentially closed fields.

In a sense, the closer analogy is to the theory  $\mathbf{ACF}_0$ . This is not entirely surprising, since differentially closed fields follow the paradigm of algebraically closed field closely in many respects. For  $\mathbf{ACF}_0$ , Theorem 4.1 is trivially true, since *every* countable algebraically closed field (indeed of arbitrary characteristic) has a computable presentation. All those of finite transcendence degree over  $\mathbb{Q}$  are relatively computably categorical, meaning that a presentation of degree  $\mathbf{d}$  has a  $\mathbf{d}$ -computable isomorphism onto a computable copy. The unique countable model of  $\mathbf{ACF}_0$  of infinite transcendence degree over  $\mathbb{Q}$  is not relatively computably categorical, but it is relatively  $\Delta_2^0$ -categorical: in one jump over the atomic diagram of the structure, one can compute a transcendence basis for the field over  $\mathbb{Q}$ . (Mapping this basis onto a basis in a computable copy is easy, given a  $\mathbf{0}'$ -oracle, and extending the isomorphism to the rest of the field requires only the atomic diagram.) For models of  $\mathbf{ACF}_0$ , one can give a substantially simplified version of the priority construction used in Theorem 4.1, guessing effectively at the minimal polynomial (possibly 0, for a transcendental) of the element  $x_n$  over the subfield  $\mathbb{Q}(x_0, \dots, x_{n-1})$  of  $K$  and building  $F$  effectively, again with an isomorphism  $h = \lim_s h_s$  computable in  $\mathbf{0}'$ . For readers who find the construction in the proof of Theorem 4.1 daunting, carrying out this construction for  $\mathbf{ACF}_0$  might be a useful prelude.

## 5 Spectra of Differentially Closed Fields

Theorem 4.1 relativizes, giving the following result.

**Proposition 5.1** *For each countable model  $K$  of  $\mathbf{DCF}_0$ , of any Turing degree  $\mathbf{c}$ , every degree  $\mathbf{d}$  with  $\mathbf{d}' \geq \mathbf{c}'$  lies in the spectrum of  $K$ .*

*Proof.* One simply runs the same construction as in Theorem 4.1, relative to an oracle from  $\mathbf{d}$ . Since  $\mathbf{d}' \geq \mathbf{c}'$ , this oracle can compute all the necessary approximations to facts about  $K$  and about minimal differential polynomials in  $K$ , so this produces a  $\mathbf{d}$ -computable differential field isomorphic to  $K$ . As mentioned in Subsection 1.3, Knight's theorem from [12] then shows that  $\mathbf{d} \in \text{Spec}(K)$ , since no differentially closed field is automorphically trivial. ■

**Definition 5.2** Let *first-jump equivalence* be the equivalence relation  $\sim_1$  on Turing degrees defined by:

$$\mathbf{c} \sim_1 \mathbf{d} \iff \mathbf{c}' = \mathbf{d}'.$$

We say that a spectrum  $\mathcal{S}$  *respects*  $\sim_1$  if, for all degrees  $\mathbf{c}$  and  $\mathbf{d}$ ,

$$\mathbf{c} \sim_1 \mathbf{d} \implies (\mathbf{c} \in \mathcal{S} \iff \mathbf{d} \in \mathcal{S}).$$

Proposition 5.1 shows that every spectrum of a model  $K$  of  $\mathbf{DCF}_0$  respects  $\sim_1$ . It follows that such a spectrum  $\text{Spec}(K)$  is actually determined by the *jump spectrum*  $\{\mathbf{d}' : \mathbf{d} \in \text{Spec}(K)\}$  of  $K$ . Moreover, this proposition, along with an easy computability-theoretic lemma, yields a quick proof and strengthening of a property for  $\mathbf{DCF}_0$  which was already known to hold for countable linear orders, Boolean algebras, and trees (viewed as partial orders), by results of Richter in [22]. This result had already been established in unpublished work by Uri Andrews and Antonio Montalbán, following the method of Richter.

**Corollary 5.3 (cf. Andrews & Montalbán)** *No countable differentially closed field  $K$  of characteristic 0 intrinsically computes any noncomputable set  $B \subseteq \omega$ . That is, the spectrum of  $K$  cannot be contained within the upper cone  $\{\mathbf{d} : \mathbf{b} \leq \mathbf{d}\}$  above a nonzero degree  $\mathbf{b}$ . In particular, if such a spectrum has a least degree under  $\leq_T$  among its elements, then that degree is  $\mathbf{0}$ .*

*Proof.* Let  $\mathbf{c}$  be the degree of  $K$ , where  $\text{Spec}(K)$  respects  $\sim_1$  and is not a singleton. Knight's theorem shows that  $\text{Spec}(K)$  must then be upwards-closed under  $\leq_T$ . (By Proposition 5.1, every countable model of  $\mathbf{DCF}_0$  has these properties.) With  $\mathbf{b} \neq \mathbf{0}$ , Lemma 5.4 below yields a degree  $\mathbf{d}$  with  $\mathbf{b} \not\leq \mathbf{d}$  and  $\mathbf{c}' \leq \mathbf{d}'$ . Now by the Relativized Friedberg Completeness Criterion (see [27, Thm. VI.3.2]), some degree  $\geq \mathbf{c}$  must have jump equal to  $\mathbf{d}'$ , and the upwards-closed set  $\text{Spec}(K)$  must contain this degree. But then  $\mathbf{d} \in \text{Spec}(K)$ , because  $\text{Spec}(K)$  respects  $\sim_1$ . Thus  $\text{Spec}(K)$  contains a degree outside the upper cone above  $\mathbf{b}$ . ■

Indeed, this corollary would hold of any set  $\mathcal{S}$  of degrees which is upwards-closed and nonempty and respects  $\sim_1$ , whether or not  $\mathcal{S}$  is a spectrum. The proof of the requisite lemma runs along the lines of the constructions in [27, Chapter VI], but does not actually appear there, so we prove it here.



**Lemma 5.4 (Folklore)** *For every noncomputable set  $B$  and every set  $C$ , there exists some set  $D$  with  $B \not\leq_T D$  and  $C' \leq_T D'$ . Indeed  $C'$  lies below the join  $\emptyset' \oplus D$ .*

*Proof.* We build finite initial segments  $\gamma_s$  of  $D$ , with  $\gamma_0$  empty and each  $\gamma_s \subseteq \gamma_{s+1}$ . Given  $\gamma_{2e}$ , let  $\gamma_{2e+1} = \gamma_{2e} \widehat{\chi_{C'}(e)}$ . (Here  $\chi_{C'}$  is the characteristic function of  $C'$ .) Next, if there exist  $\sigma, \tau \in 2^{<\omega}$  and  $x, t \in \omega$  with  $\gamma_{2e+1} \subseteq \sigma$  and  $\gamma_{2e+1} \subseteq \tau$  and  $\Phi_{e,t}^\sigma(x) \downarrow \neq \Phi_{e,t}^\tau(x) \downarrow$ , then take the least such 4-tuple  $\langle \sigma, \tau, x, t \rangle$  (in some fixed enumeration of  $((\omega^{<\omega})^2) \times \omega^2$ ), and choose either  $\gamma_{2e+2} = \sigma$  if  $\Phi_e^\tau(x) = \chi_B(x)$ , or  $\gamma_{2e+2} = \tau$  otherwise. If no such triple exists, we keep  $\gamma_{2e+2} = \gamma_{2e+1}$ . Set  $D = \bigcup_n \gamma_n$ .

Our choice of each  $\gamma_{2e+2}$  ensures that  $\Phi_e^D \neq \chi_B$ . This is clear if we found a 4-tuple as described, so assume no such 4-tuple existed. Then either  $\Phi_e^D$  is not total, or else, without any oracle, we may compute  $\Phi_e^D(x)$  for each  $x$ : just find any  $\sigma \supseteq \gamma_{2e+1}$  and any  $t$  for which  $\Phi_{e,t}^\sigma(x) \downarrow$ , and output its value. Totality of  $\Phi_e^D$  ensures that we will find such a  $\sigma$  and  $t$ , and if  $\Phi_{e,t}^\sigma(x) \downarrow \neq \Phi_e^D(x)$ , then a 4-tuple would have existed. Since  $B >_T \emptyset$ , this proves that  $\chi_B$  cannot equal the computable function  $\Phi_e^D$ . But with  $\Phi_e^D \neq \chi_B$  for all  $e$ , we have  $B \not\leq_T D$ .

Now, to determine whether  $e \in C'$  using oracles for  $\emptyset'$  and  $D$  (or alternatively, from a  $D'$ -oracle), we need only compute the length  $|\gamma_{2e}|$  of  $\gamma_{2e}$  and then check whether  $|\gamma_{2e}| \in D$ . For  $e = 0$ ,  $|\gamma_{2e}| = 0$ , so assume recursively that we have computed the length  $n$  of  $\gamma_{2e-2}$ , and hence  $|\gamma_{2e-1}| = (n+1)$ . With our  $D$ -oracle, this is equivalent to having computed  $\gamma_{2e-1}$  itself, because  $\gamma_{2e-1} = D \upharpoonright (n+1)$ . Now use the  $\emptyset'$ -oracle to determine whether any 4-tuple  $\langle \sigma, \tau, x, t \rangle$  existed satisfying the condition above for  $\gamma_{2e-1}$ . If not, then  $\gamma_{2e} = \gamma_{2e-1}$ . If it does exist, we search and find the least such 4-tuple. Notice that  $\sigma \not\subseteq \tau$  and  $\tau \not\subseteq \sigma$ , because it would be impossible to have  $\Phi_e^\sigma(x) \downarrow \neq \Phi_e^\tau(x) \downarrow$  if either were an initial segment of the other. But this means that  $\sigma \subseteq \chi_D$  if and only if  $\tau \not\subseteq \chi_D$ . So, by checking which one is compatible with  $\chi_D$ , we may decide whether  $\gamma_{2e}$  equals  $\sigma$  or  $\tau$ , and thus we have computed  $\gamma_{2e}$  from our  $D$ -oracle and thereby determined  $\chi_{C'}(e) = \chi_D(|\gamma_{2e}|)$ .  $\blacksquare$

The main consequence of Proposition 5.1 is a very precise description of the spectra of models of  $\mathbf{DCF}_0$  in terms of arbitrary spectra.

**Theorem 5.5** *For a set  $\mathcal{S}$  of Turing degrees, the following are equivalent.*

1.  $\mathcal{S}$  is the spectrum of some countable model  $K$  of  $\mathbf{DCF}_0$ .

2. There exists a countable, automorphically nontrivial graph  $G$  with

$$\mathcal{S} = \{\mathbf{d} : \mathbf{d}' \in \text{Spec}(G)\}.$$

3.  $\mathcal{S}$  respects  $\sim_1$  and there exists a countable, automorphically nontrivial graph  $J$  with  $\mathcal{S} = \text{Spec}(J)$ .

Theorem 1.9 shows that items (2) and (3) here could equally well allow  $G$  and  $J$  to vary over all countable, automorphically nontrivial structures in all computable languages.

*Proof.* The implication (2)  $\implies$  (1) is precisely Corollary 3.3 above. Also, (1)  $\implies$  (3) follows from Proposition 5.1 and Theorem 1.9. To prove (3)  $\implies$  (2), given  $J$ , we appeal to the following theorem, proven independently by Montalbán in [18] and by Soskova and Soskov in [28].

**Theorem 5.6** (see [18, 28]) *For every countable structure  $\mathcal{A}$ , there exists another countable structure  $\mathcal{A}'$ , the jump of the structure  $\mathcal{A}$ , such that*

$$\text{Spec}(\mathcal{A}') = \{\mathbf{c}' : \mathbf{c} \in \text{Spec}(\mathcal{A})\}.$$

Using Theorem 1.9, we may convert the jump  $J'$  of our  $J$  into another graph  $G$ , with  $\text{Spec}(G) = \{\mathbf{c}' : \mathbf{c} \in \text{Spec}(J)\}$ . Theorem 5.6 shows this  $G$  to be automorphically nontrivial, since its spectrum is certainly not a singleton. So each  $\mathbf{d} \in \mathcal{S} = \text{Spec}(J)$  has  $\mathbf{d}' \in \text{Spec}(G)$ . Conversely, for every degree  $\mathbf{d}$  with  $\mathbf{d}' \in \text{Spec}(G)$ , there exists  $\mathbf{c} \in \text{Spec}(J) = \mathcal{S}$  with  $\mathbf{c}' = \mathbf{d}'$ , and then  $\mathbf{d} \in \mathcal{S}$  because  $\mathcal{S}$  respects  $\sim_1$ . ■

## References

- [1] C.J. Ash, C.G. Jockusch, Jr., & J.F. Knight; Jumps of orderings, *Trans. Amer. Math. Soc.* **319** (1990) 2, 573–599.
- [2] L. Blum; *Generalized Algebraic Theories: A Model Theoretic Approach*, Ph.D. thesis, Massachusetts Institute of Technology, 1968.
- [3] L. Blum; Differentially closed fields: a model-theoretic tour. in *Contributions to algebra* (collection of papers dedicated to Ellis Kolchin), eds. H. Bass, P. Cassidy, & J. Kovacic (New York: Academic Press, 1977), 37–61.

- [4] R.G. Downey & C.G. Jockusch, Jr.; Every low Boolean algebra is isomorphic to a recursive one, *Proc. Amer. Math. Soc.* **122** (1994), 871–880.
- [5] R. Downey & J.F. Knight; Orderings with  $\alpha$ th jump degree  $\mathbf{0}^{(\alpha)}$ , *Proc. Amer. Math. Soc.* **114** (1992) 2, 545–552.
- [6] A. Frolov, V. Harizanov, I. Kalimullin, O. Kudinov, & R. Miller; Degree spectra of  $\text{high}_n$  and  $\text{non-low}_n$  degrees, *Journal of Logic and Computation* **22** (2012) 4, 755–777.
- [7] L. Harrington; Recursively presentable prime models, *Journal of Symbolic Logic* **39** (1974) 2, 305–309.
- [8] D.R. Hirschfeldt, B. Khoussainov, R.A. Shore, & A.M. Slinko; Degree spectra and computable dimensions in algebraic structures, *Ann. Pure Appl. Logic* **115** (2002), 71–113.
- [9] E. Hrushovski & M. Itai, On model complete differential fields, *Trans. Amer. Math. Soc.* **355** (2003), 11, 4267–4296
- [10] E. Hrushovski & Z. Sokolović, Minimal subsets of differentially closed fields, preprint from the early 1990s.
- [11] C.G. Jockusch & R.I. Soare; Degrees of orderings not isomorphic to recursive linear orderings, *Ann. Pure Appl. Logic* **52** (1991), 39–64.
- [12] J.F. Knight; Degrees coded in jumps of orderings, *Journal of Symbolic Logic* **51** (1986), 1034–1042.
- [13] J.F. Knight & M. Stob; Computable Boolean algebras, *Journal of Symbolic Logic* **65** (2000) 4, 1605–1623.
- [14] D. Marker, Manin kernels, *Connections between model theory and algebraic and analytic geometry*, 1–21, Quad. Mat., 6, Dept. Math., Seconda Univ. Napoli, Caserta, 2000.
- [15] D. Marker; Model theory of differential fields, in *Model Theory of Fields*, eds. D. Marker, M. Messmer, & A. Pillay, vol. 5 in the *ASL Lecture Notes in Logic* (Wellesley, MA: A.K. Peters, Ltd., 2006), pp. 41–109.

- [16] R. Miller, A. Ovchinnikov, & D. Trushin; Computing constraint sets for differential fields, to appear in the *Journal of Algebra*.
- [17] R. Miller, J. Park, B. Poonen, H. Schoutens, & A. Shlapentokh; A computable functor from graphs to fields, to appear.
- [18] A. Montalbán; Notes on the jump of a structure, *Mathematical Theory and Computational Practice* 2009, 372–378.
- [19] J. Nagloo & A. Pillay, On algebraic relations between solutions of a generic Painlevé equation, preprint.
- [20] A. Pillay; Differential fields, in *Lectures on algebraic model theory*, Fields Inst. Monogr., 15 (Providence, RI: Amer. Math. Soc., 2002) 1–45.
- [21] A. Pillay, Differential algebraic groups and the number of countable differentially closed fields, in *Model Theory of Fields*, eds. D. Marker, M. Messmer, & A. Pillay, vol. 5 in the *ASL Lecture Notes in Logic* (Wellesley, MA: A.K. Peters, Ltd., 2006), pp. 111–133.
- [22] L.J. Richter; Degrees of structures, *Journal of Symbolic Logic* **46** (1981), 723–731.
- [23] J.F. Ritt; *Differential Equations from the Algebraic Standpoint*, AMS Colloquium publications, vol. XIV (New York: Amer. Math. Soc., 1932).
- [24] G.E. Sacks; *Saturated Model Theory* (Reading: W.A. Benjamin, 1972).
- [25] S. Shelah, L. Harrington, & M. Makkai; A proof of Vaught’s conjecture for  $\omega$ -stable theories, *Israel J. Math.* **49** (1984) 1–3, 259–280.
- [26] T. Slaman; Relative to any nonrecursive set, *Proc. Amer. Math. Soc.* **126** (1998), 2117–2122.
- [27] R.I. Soare; *Recursively Enumerable Sets and Degrees* (New York: Springer-Verlag, 1987).
- [28] A.A. Soskova & I.N. Soskov; A jump inversion theorem for the degree spectra, *Journal of Logic and Computation* **19** (2009) 1, 199–215
- [29] J.J. Thurber; Every low<sub>2</sub> Boolean algebra has a recursive copy, *Proc. Amer. Math. Soc.* **123** (1995), 3859–3866.

- [30] S. Wehner; Enumerations, countable structures, and Turing degrees,  
*Proc. Amer. Math. Soc.* **126** (1998), 2131–2139.

DEPARTMENT OF MATHEMATICS, STATISTICS, & COMPUTER  
SCIENCE

UNIVERSITY OF ILLINOIS AT CHICAGO M/C 249

851 S. MORGAN ST.

CHICAGO, IL 60607 U.S.A.

*E-mail:* marker@math.uic.edu

DEPARTMENT OF MATHEMATICS  
QUEENS COLLEGE – C.U.N.Y.  
65-30 KISSENA BLVD.

FLUSHING, NEW YORK 11367 U.S.A.

PH.D. PROGRAMS IN MATHEMATICS & COMPUTER SCIENCE  
C.U.N.Y. GRADUATE CENTER

365 FIFTH AVENUE

NEW YORK, NEW YORK 10016 U.S.A.

*E-mail:* Russell.Miller@qc.cuny.edu